Exam in Linear Algebra

First Year at The Faculties of Engineering and Science and of Health

January 3rd, 2017, 9.00-13.00

This test has 10 pages and 15 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is **not** allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: ______________________________

STUDENT NUMBER: ______________________________

COURSE:

☐ Hold 1 (Jacob Broe)
☐ Hold 2 (Nikolaj Hess-Nielsen)
☐ English course (Athanasios Georgiadis)

In all problems: *there is only one correct answer to each question.*

Problem 1 (5%)

What is the number of solutions of the following system of linear equations

$$\begin{aligned} x_1 + x_2 \quad\;\; &= 2 \\ 2x_1 + x_2 + x_3 &= 3 \\ x_1 \quad\;\; + x_3 &= 0 \end{aligned}$$

☒ 0
☐ 1
☐ infinitely many

Problem 2 (5%)

Let A be a $4 \times n$ matrix and let E be the elementary matrix $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

How does the matrix EA appear from A?

☐ By adding 3 times row 1 to row 3.
☐ By adding 3 times row 3 to row 1.
☐ By adding -3 times row 3 to row 1.
☒ By adding -3 times row 1 to row 3.
☐ By adding 3 times column 1 to column 3.
☐ By adding -3 times column 3 to column 1.

Problem 3 (10%)

Let $T : \mathcal{R}^n \to \mathcal{R}^m$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 & 4 \\ 1 & 1 & 0 & 2 & 2 \end{bmatrix}.$$

1. What is the value of n ?

   ☐ 2 ☐ 3 ☐ 4 ☒ 5 ☐ 6

2. What is the value of m ?

   ☐ 2 ☒ 3 ☐ 4 ☐ 5 ☐ 6

3. What the rank of A ?

   ☐ 2 ☒ 3 ☐ 4 ☐ 5

4. What is the dimension of the null space of T ?

   ☐ 0 ☐ 1 ☒ 2 ☐ 3 ☐ 4 ☐ 5

5. Is T one-to-one?

   ☐ Yes ☒ No

6. Is T onto?

   ☒ Yes ☐ No

Problem 4 (6%)

Let $\mathbf{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$. Then $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\mathcal{R}^3$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the orthogonal basis for $\mathcal{R}^3$ obtained by using the Gram-Schmidt process on $\mathcal{B}$. Then $\mathbf{v}_1 = \mathbf{u}_1$.
What is $\mathbf{v}_2$?

☐ $\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ -1\end{bmatrix}$ ☐ $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ ☒ $\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ 1\end{bmatrix}$ ☐ $\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$

Problem 5 (6%)

Let A be an $m \times n$ matrix and let $B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\ \mathbf{b}_4\ \mathbf{b}_5]$ and C be matrices satisfying that the products AB, BC and CA are defined.

1. How many columns are there in AB ?

   ☒ 5 ☐ m ☐ n

2. What is the size of BC ?

   ☐ $m \times n$ ☐ $m \times 5$ ☐ $5 \times n$ ☒ $n \times m$

Problem 6 (10%)

The characteristic polynomial of

$$A = \begin{bmatrix} -4 & 6 & -6 & 6 \\ -1 & 3 & -2 & 2 \\ -1 & 1 & 0 & 2 \\ -3 & 3 & -3 & 5 \end{bmatrix}$$

is $(t-1)(t+1)(t-2)^2$.

1. Let $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. For which value of λ is $A\mathbf{v} = \lambda\mathbf{v}$?

   ☐ 1 ☐ -1 ☒ 2 ☐ -2

2. Which one of the following is an eigenvector of A?

   ☐ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ☒ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

3. Is A invertible?

   ☒ Yes ☐ No

4. Is A diagonalizable?

   ☒ Yes ☐ No

Problem 7 (6%)

Let T be the linear transformation with standard matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$ is a basis for $\mathcal{R}^2$.

Which one of the following is the matrix representation of T with respect to $\mathcal{B}$, denoted by $[T]_{\mathcal{B}}$?

☐ $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$ ☒ $\begin{bmatrix} -2 & -4 \\ 3 & 7 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 3 \\ 4 & 10 \end{bmatrix}$ ☐ $\begin{bmatrix} -2 & 6 \\ -1 & 4 \end{bmatrix}$

Problem 8 (10%)

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, let $W = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$ and let $\mathbf{u} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$.

1. Are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ orthogonal?

   ☒ Yes ☐ No

2. What is the orthogonal projection of $\mathbf{u}$ on W ?

   ☐ $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$ ☒ $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

3. What is the orthogonal projection of $\mathbf{u}$ on $W^\perp$?

   ☐ $\begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$ ☒ $\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$

4. What is the dimension of $W^\perp$?

   ☐ 0 ☒ 1 ☐ 2 ☐ 3 ☐ 4

Problem 9 (8%)

Let $A = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 4 & 1 & -2 \\ 2 & 1 & 1 & -1 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

1. Is $\mathbf{b}$ contained in Col A? ☐ Yes ☒ No
2. Is $\mathbf{c}$ contained in Col A? ☐ Yes ☒ No
3. Is $\mathbf{b}$ contained in Null A? ☐ Yes ☒ No
4. Is $\mathbf{c}$ contained in Null A? ☒ Yes ☐ No

Problem 10 (6%)

Let A and B be 7×7 matrices with $\det A = 5$ and $\det B = 3$.

1. What is $\det(-A)$?

   ☒ -5 ☐ 5 ☐ 0 ☐ 4

2. What is $\det A^T B$?

   ☐ -15 ☐ -2 ☐ 2 ☒ 15 ☐ $\frac{5}{3}$ ☐ $\frac{3}{5}$ ☐ $-\frac{3}{5}$

3. What is $\det A^{-1} B$?

   ☐ -15 ☐ -2 ☐ 2 ☐ 15 ☐ $\frac{5}{3}$ ☒ $\frac{3}{5}$ ☐ $-\frac{3}{5}$

Problem 11 (4%)

Let $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 1 & 1 \\ 0 & -2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$.

Let $C = AB$. What is the $(2,1)$-entry in C, i.e., c_{21} ?

☐ -5 ☐ -4 ☐ -3 ☐ 3 ☐ 4 ☒ 5

Problem 12 (4%)

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 5 \end{bmatrix}$.

What is the determinant of A?

☐ -10 ☐ -8 ☐ -4 ☒ 4 ☐ 8 ☐ 10

Problem 13 (5%)

Let $Q = c \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$, where a and c are constants.

For which combination of a and c is Q an orthogonal matrix ?

☐ $a = 2, c = \frac{1}{\sqrt{12}}$
☐ $a = -1, c = 3$
☐ $a = 0, c = \sqrt{8}$
☐ $a = 1, c = 3$
☒ $a = -1, c = \frac{1}{3}$
☐ $a = 1, c = -\frac{1}{3}$

Problem 14 (9%)

Let A be a 12×15 matrix.
Answer the following true/false problems about A.

1. A is a square matrix ☐ True ☒ False
2. Col A is a subspace of $\mathcal{R}^{12}$ ☒ True ☐ False
3. Col A is a subspace of $\mathcal{R}^{15}$ ☐ True ☒ False
4. Col A and Row A have the same dimension ☒ True ☐ False
5. Col A and Null A have the same dimension ☐ True ☒ False

Problem 15 (6%)

The following commands are entered in the MATLAB Command Window:

```
>> u = [1; 1; 1; 1];
>> v = [1; 2; 3; 4];
>> w = [1; 3; 6; 10];
>> x = [1; 4; 10; 19 ];
>> A = [u v w x];
>> rref(A)

ans =

     1     0     0     1
     0     1     0    -3
     0     0     1     3
     0     0     0     0

>> det(A)
```

1. Which one of the following is true?

   ☐ v is a row vector
   ☒ v is a column vector
   ☐ v is a 2×2 matrix

2. What is MATLAB's answer to the last command?

   ☐ -3
   ☐ -9
   ☐ 3
   ☒ 0
   ☐ 1