Exam in Linear Algebra

First Year at The Faculties of Engineering and Science and of Health

June 6th, 2016, 9.00-13.00

This test has 10 pages and 15 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is **not** allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: ______________________________

STUDENT NUMBER: ______________________________

COURSE:

☐ Aalborg Hold 5 (Jacob Broe)
☐ Aalborg Hold 6 (Nikolaj Hess-Nielsen)
☐ Esbjerg Dansk hold (Ulla Tradsborg)
☐ Esbjerg, English course (Johnny Weile)

In all problems: *there is only one correct answer to each question.*

Problem 1 (6 %)

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$ be a matrix with 3 rows and let $B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3]$ be such that $C = AB$ is defined.

1. How many rows are there in the matrix B?

   ☐ 2
   ☐ 3
   ☒ 4
   ☐ The number of rows in B can not be determined from the given information.

2. How many rows are there in the matrix C?

   ☐ 2
   ☒ 3
   ☐ 4
   ☐ The number of rows in C can not be determined from the given information.

3. How can the second column in $C = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3]$ be computed?

   ☐ $\mathbf{c}_2 = A\mathbf{a}_2$ ☐ $\mathbf{c}_2 = B\mathbf{a}_2$ ☒ $\mathbf{c}_2 = A\mathbf{b}_2$ ☐ $\mathbf{c}_2 = B\mathbf{b}_2$

Problem 2 (4 %)

Let A be a $3 \times n$ matrix and let E be the elementary matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

How does the matrix EA appear from A?

☐ By adding 2 times row 1 to row 3.
☐ By adding 2 times row 2 to row 3.
☒ By adding 2 times row 3 to row 2.
☐ By adding 2 times column 1 to column 3.
☐ By adding 2 times column 2 to column 3.
☐ By adding 2 times column 3 to column 2.

Problem 3 (10 %)

Let $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. The matrix $\begin{bmatrix} 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix}$ has the following reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

1. Answer the following problems about pivot columns of A:

   column 1 is a pivot column. ☒ True ☐ False

   column 2 is a pivot column. ☒ True ☐ False

   column 3 is a pivot column. ☐ True ☒ False

   column 4 is a pivot column. ☒ True ☐ False

2. What is the nullity of A?

   ☐ 0 ☒ 1 ☐ 2 ☐ 3 ☐ 4

3. Let $\mathbf{x}$ be a solution of $A\mathbf{x} = \mathbf{b}$. What is x_4 ?

   ☐ 1 ☐ 3
   ☒ 2 ☐ x_4 is a free variabel.

4. Answer the following true/false problems:
   Every solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ satisfies $x_1 = x_3$.

   ☒ True ☐ False

   The set of solutions of $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathcal{R}^4$

   ☐ True ☒ False

Problem 4 (10 %)

Let $\mathbf{q}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\mathbf{q}_2 = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$, $\mathbf{q}_3 = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$, $\mathbf{q}_4 = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$,

$A = \begin{bmatrix}2 & 1 & -1 & 0\\1 & 2 & 0 & -1\\-1 & 0 & 2 & 1\\0 & -1 & 1 & 2\end{bmatrix}$, and let $Q = [\mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3\ \mathbf{q}_4]$.

We see that $\mathcal{B} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ is an orthonormal basis of $\mathcal{R}^4$.

1. Answer the following true/false problems about the matrix Q.

Q is an orthogonal matrix.	☒ True	☐ False
Q is a symmetric matrix.	☒ True	☐ False
$Q^{-1} = -Q$	☐ True	☒ False
$Q^{-1} = Q$	☒ True	☐ False

2. The determinant of Q is one of the following numbers. Which one?

   ☐ -3 ☒ -1 ☐ 0 ☐ 2 ☐ 5

3. T is now the linear operator with standard matrix A. Let $C = [T]_{\mathcal{B}}$ be the matrix representation of T with respect to the basis $\mathcal{B}$. What is c_{11}?

   ☐ 0 ☐ 1 ☒ 2 ☐ 4 ☐ 8

Problem 5 (10 %)

The characteristic polynimial of

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 2 \\ -1 & 0 & 2 & 1 \end{bmatrix}$$

is $t(t-2)(t+2)(t-4)$.

1. Which one of the following is an eigenvalue of A?

   ☐ 1 ☒ 4 ☐ -4 ☐ 16

2. Which one of the following is an eigenvector of A?

   ☐ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ ☒ $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

3. Is A diagonalizable?

   ☒ Yes ☐ No

4. Is A invertible?

   ☐ Yes ☒ No

Problem 6 (10 %)

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}$ and let $W = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 3 \end{bmatrix}$ and let $\mathbf{w}$ be the orthogonal projection of $\mathbf{u}$ on W.

1. Are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ orthogonal?

   ☒ Yes ☐ No

2. What is the second component of $\mathbf{w}$ (i.e. w_2)?

   ☐ -4 ☐ -1 ☐ 0 ☐ 1 ☒ 4 ☐ 9

3. Let $\mathbf{z}$ be the orthogonal projection of $\mathbf{u}$ on $W^{\perp}$. What is the second component of $\mathbf{z}$ (i.e. z_2)?

   ☐ -4 ☒ -1 ☐ 0 ☐ 1 ☐ 4 ☐ 9

4. What is the dimension of $W^{\perp}$?

   ☐ 0 ☐ 1 ☒ 2 ☐ 3 ☐ 4 ☐ 5

Problem 7 (4 %)

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Which one of the following statements is true:

☒ $A = A_{90^\circ}$
☐ $A = A_{180^\circ}$
☐ $A = A_{270^\circ}$
☐ $A = A_\theta$ for some other angle θ
☐ A is not a rotation matrix.

Problem 8 (4 %)

Let $A = \begin{bmatrix} 7 & 9 & -6 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Let $C = AB$. What is the number c_{23} ?

☐ -7 ☐ -2 ☐ 1 ☒ 10 ☐ 13

Problem 9 (6 %)

Let A and B be 5×5 matrices with $\det A = 3$ and $\det B = 2$.

1. What is $\det(-2A)$?

   ☐ -486 ☒ -96 ☐ -6 ☐ 6 ☐ 96 ☐ 486

2. What is $\det AB^T$?

   ☐ -9 ☐ -6 ☐ -5 ☐ 5 ☒ 6 ☐ 9

3. What is $\det AB^{-1}$?

   ☐ 1 ☐ $\frac{1}{6}$ ☒ $\frac{3}{2}$ ☐ $\frac{2}{3}$ ☐ 6 ☐ $-\frac{1}{6}$

Problem 10 (4 %)

Let $A = \begin{bmatrix} -1 & 2 & 3 \\ -1 & 2 & 4 \\ 2 & 1 & 4 \end{bmatrix}$.

What is the determinant of A?

☐ -8 ☐ -5 ☐ -2 ☐ 2 ☒ 5 ☐ 8

Problem 11 (10 %)

$T : \mathcal{R}^n \to \mathcal{R}^m$ is a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 2 & 0 \\ -1 & 2 & 1 & 0 & 2 & 1 \end{bmatrix}.$$

Using elementary row operations A can be transformed into

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

1. What is the value of n?

   ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☒ 6 ☐ 7

2. What is the value of m?

   ☐ 2 ☒ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7

3. What is the rank of A?

   ☐ 2 ☒ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7

4. What is the dimension of the null space of T?

   ☐ 0 ☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5

5. Is T one-to-one?

   ☐ Yes ☒ No

6. Is T onto?

   ☒ Yes ☐ No

Problem 12 (7 %)

Let $A = \begin{bmatrix} 1 & -6 & 7 \\ 2 & -5 & 8 \\ 3 & -4 & 9 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$.

1. Is $\mathbf{b}$ contained in Col A? ☒ Yes ☐ No
2. Is $\mathbf{b}$ contained in Null A? ☐ Yes ☒ No
3. Is $\mathbf{b}$ contained in $(\text{Row } A)^{\perp}$? ☐ Yes ☒ No

Problem 13 (5 %)

What is the number of solutions of the following system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 \phantom{{}+x_2} - x_3 &= 0 \\ 2x_1 + 3x_2 + 4x_3 &= 9 \end{aligned}$$

☐ 0
☐ 1
☐ 2
☒ infinitely many.

Problem 14 (4 %)

$T : \mathcal{R}^5 \to \mathcal{R}^3$ is a linear transformation.

1. What is the smallest possible dimension of the null space of T?

   ☐ 0 ☐ 1 ☒ 2 ☐ 3 ☐ 4 ☐ 5

2. What is the largest possible dimension of the null space of T?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☒ 5

Problem 15 (6 %)

The following matrix has been entered in the MATLAB Command Window:

```
>> A = [1 1 1; 1 1 2; 1 2 2];
```

It is known that `A` has an inverse matrix `B`. Which one of the following combinations of MATLAB commands computes the correct inverse of `A` ?

☐ `C=rref([A eye(3)]); B=C(4:6,:)`
☐ `C=rref([eye(3) A]); B=C(:,1:3)`
☐ `C=rref([eye(3) A]); B=C(1:3,:)`
☐ `C=rref([A eye(3)]); B=C(:,1:3)`
☒ `C=rref([A eye(3)]); B=C(:,4:6)`
☐ `C=rref([eye(3) A]); B=C(:,4:6)`