Exam in Discrete Mathematics

First Year at The TEK-NAT Faculty

August 19th, 2014, 9.00–13.00

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam has two independent parts.

- Part I contains ”regular exercises”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II contains ”multiple choice”exercises. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each page of your answers. **Number each page and write the total number of pages on the front page of the answers.**

NAME: ______________________________

STUDENT NUMBER: ______________________________

Part I (”regular exercises”)

Exercise 1 (6%).

Find the expansion of $(a+1)^7$ using The Binomial Theorem.

Exercise 2 (9%).

Let $A = \{a, b, c, d, e\}$ and let $R = \{(a, b), (b, c), (c, a), (c, d), (c, e), (d, e)\}$ be a relation on A.

1. Draw the directed graph representing R.
2. Determine a matrix $\mathbf{M}_R$ representing R.
3. Determine the transitive closure R^* of R.

Exercise 3 (7%).

Compute

$$(703 \cdot 7004 + 70005) \mod 7.$$

Exercise 4 (9%).

Prove by induction that

$$\sum_{i=1}^{n}(6i-4) = 3n^2 - n,$$

for every positive integer n.

Exercise 5 (7%).

Use the Euclidean algorithm to determine the greatest common divisor of 161 and 91, and find integers s and t satisfying that $\gcd(161, 91) = s \cdot 161 + t \cdot 91$.

Exercise 6 (8%).

Find witnesses proving that $f(x) = 3x^2 + 10x + 2$ is $O(x^2)$.

Exercise 7 (12%).

Consider the following algorithm:

Procedure exam(n: positive integer)
$x := 4$
$y := 12$
$i := 1$
while $i < n$
    $h := 4(y - x)$
    $x := y$
    $y := h$
    $i := i + 1$
return y

1. Show that the following proposition is an invariant of the while-loop:

$$i \in \mathbb{N} \ \wedge \ i \leq n \ \wedge \ x = (i+1) \cdot 2^i \ \wedge \ y = (2i+4) \cdot 2^i.$$

2. What is the value of y when the loop terminates? Justify your answer.

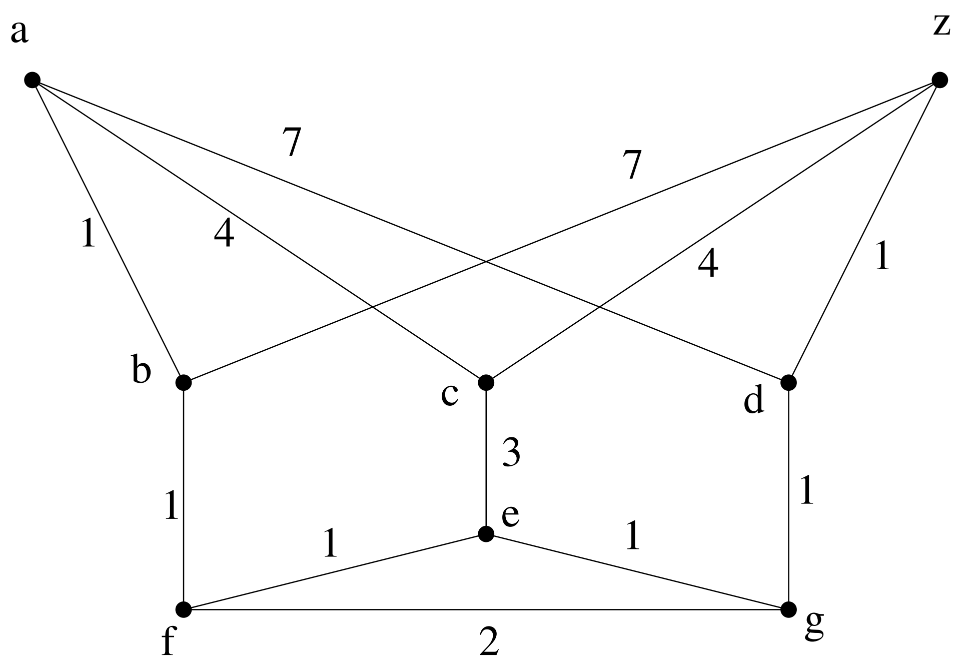

Figure 1: A graph G considered in Exercise 8 and Exercise 10.

Exercise 8 (12%).

Use Dijkstra's algorithm to determine the length of a shortest path from **a** to **z** in the graph shown in Figure 1.

Part II ("multiple choice" exercises)

Exercise 9 (4%).

Consider the following algorithm:

Procedure sum(n: positive integer)
$s := 0$
for $i := 1$ **to** n
  for $j := 1$ **to** n
    $s := s + 1$
return s

The worst-case time complexity of procedure sum is:

☐ $O(n)$ ☐ $O(n \log n)$ ☐ $O(n^2)$ ☐ $O(n^3)$

Exercise 10 (4%).

Consider the graph G in Figure 1. (In this exercise we may disregard the edge-weights of G.) Answer the following two true/false exercises:

1. G has a Hamilton circuit.

   ☐ True ☐ False

2. G has an Euler circuit.

   ☐ True ☐ False

Exercise 11 (6%).

Let T be a full binary tree with 7 leaves.

1. How many vertices are there in T in all?

   ☐ 6 ☐ 7 ☐ 13 ☐ 14 ☐ 15

2. What is least possible height of T?

   ☐ 2 ☐ 3 ☐ 4 ☐ 5

3. What is the largest possible height of T?

   ☐ 5 ☐ 6 ☐ 7 ☐ 8

Exercise 12 (6%).

Let $A = \{1, 2\}$ and $B = \{2, 3, 4\}$ be sets.

1. What is the cardinality of $\mathcal{P}(A \times B)$?

   ☐ 12 ☐ 16 ☐ 32 ☐ 64 ☐ 128

2. Which of the following are elements of $\mathcal{P}(A \times B)$?

   ☐ $(1, 3)$ ☐ $\{1, 4\}$ ☐ $\emptyset$ ☐ $\{(1, 3), (2, 2)\}$

Exercise 13 (10%).

Let $f(x) = 2x^4 + 5x^3 + x \log x + x^2 + 2$, for $x > 0$.
Answer the following 5 true/false exercises:

1. $f(x)$ is $O(x^5)$.

   ☐ True ☐ False

2. $f(x)$ is $O(x^4)$.

   ☐ True ☐ False

3. $f(x)$ is $O(x^3)$.

   ☐ True ☐ False

4. $f(x)$ is $O(x^4 \log x)$.

   ☐ True ☐ False

5. $f(x)$ is $O(x^3 \log x)$.

   ☐ True ☐ False