Exam in Discrete Mathematics

First Year at The TEK-NAT Faculty

June 10th, 2016, 9.00-13.00

This exam consists of 11 numbered pages with 16 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: ____

STUDENT NUMBER: ____

There is only one correct answer to each question.

Problem 1 (8 %)

Consider the following linear homogeneous recurrence relation

$$a_n = -a_{n-1} + 6a_{n-2}.$$

1. What is the degree of this recurrence relation?

   ☐ -1 ☐ 0 ☒ 2 ☐ 6

2. Which of the following is the solution of this recurrence relation (α_1 and α_2 are constants)?

   ☐ $a_n = \alpha_1(-2)^n + \alpha_2 \cdot 3^n$
   ☐ $a_n = \alpha_1(-1)^n + \alpha_2 \cdot 6^n$
   ☐ $a_n = \alpha_1 + \alpha_2(-6)^n$
   ☒ $a_n = \alpha_1 \cdot 2^n + \alpha_2(-3)^n$

Problem 2 (5 %)

Consider the following algorithm:

procedure *multiplications*(n: positive integer)
$t := 1$
for $i := 1$ **to** n
  for $j := 1$ **to** i
    $t := 2 \cdot t$
return t

The number of multiplications used by this algorithm is

☐ $O(n)$ ☐ $\Theta(n)$ ☐ $O(n\sqrt{n})$ ☒ $O(n^2)$ ☐ $\Omega(n^3)$

Problem 3 (5 %)

Which one of the following numbers is an inverse of 43 modulo 100?

☐ -3 ☒ 7 ☐ 17 ☐ 27

Problem 4 (9 %)

Consider the following algorithm:

procedure *additions*(n: positive integer)
$i := 0$
$x := n$
$s := 0$
while $i < n$
  $i := i + 1$
  $s := s + i + x$
  $x := x - 1$
return s

1. Which one of the following statements is a loop invariant for the while loop in this algorithm?

   ☐ $i \leq n \quad \wedge \quad s = in \quad \wedge \quad x - i = n$
   ☐ $i \leq n - 1 \quad \wedge \quad s = i(n-1) \quad \wedge \quad i + x = n$
   ☒ $i \leq n \quad \wedge \quad s = i(n+1) \quad \wedge \quad i + x = n$
   ☐ $i \leq n - 1 \quad \wedge \quad s = i(n+1) \quad \wedge \quad x - i = n$
   ☐ $i \leq n \quad \wedge \quad s = in \quad \wedge \quad i + x = n$

2. What is the value of s returned by procedure *additions*?

   ☐ n^2 ☐ $(n+1)^2$ ☒ $n(n+1)$ ☐ $n^2 - n$

Problem 5 (6 %)

What is value of $(79 + 778 + 7777 \cdot 321) \mod 7$?

☐ 0 ☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5 ☐ 6

Problem 6 (8 %)

Let $P(n)$ be the following statement about the absolute value of real numbers

$$|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|, \text{ for all real numbers } x_1, \ldots, x_n.$$

From an exercise in the book by Rosen, we know that $P(2)$ is true.
We want to prove by induction that $P(n)$ is true for all $n \geq 2$.

Which one of the following is a correct outline of the inductive step?

☐ Let $k \geq 1$ and assume that $P(k)$ is true.
Then use $|x_1 + x_2 + x_3 \ldots + x_{k+1}| \leq |x_1 + x_2| + |x_3 + \ldots + x_{k+1}|$ and the induction hypothesis to prove $P(k+1)$.

☐ Let $k \geq 2$ and assume that $P(k)$ is true.
Then use $|x_1 + x_2 + \ldots + x_{k-1} + x_k| \leq |x_1 + x_2 + \ldots + x_{k-1}| + |x_k|$ and the induction hypothesis to prove $P(k+1)$.

☒ Let $k \geq 2$ and assume that $P(k)$ is true.
Then use $|x_1 + x_2 + \ldots + x_k + x_{k+1}| \leq |x_1 + x_2 + \ldots + x_k| + |x_{k+1}|$ and the induction hypothesis to prove $P(k+1)$.

☐ Let $k \geq 3$ and assume that $P(k)$ is true.
Then use $|x_1 + x_2 + \ldots + x_{k+1}| \leq |x_1| + |x_2 + \ldots + x_{k+1}|$ and the induction hypothesis to prove $P(k+1)$.

☐ Let $k \geq 2$ and assume that $P(k-1)$ is true.
Then use $|x_1 + x_2 + \ldots + x_k + x_{k+1}| \leq |x_1 + x_2 + \ldots + x_k| + |x_{k+1}|$ and the induction hypothesis to prove $P(k)$.

Problem 7 (9 %)

Consider the following two relations on the set $A = \{1, 2, 3, 4, 5\}$:

$$R = \{(1,1), (1,2), (2,3), (3,4), (4,5), (5,5)\}$$

$$S = \{(1,2), (1,3), (2,1), (3,1), (3,4), (3,5), (4,3), (4,4), (5,3)\}.$$

1. Answer the following true/false problems:

R is reflexive	☐ True	☒ False
R is antisymmetric	☒ True	☐ False
S is symmetric	☒ True	☐ False
S is transitive	☐ True	☒ False

2. Which one of the following values of x satisfies that $(5, x)$ is in the composed relation $R \circ S$?

   ☐ 1 ☐ 2 ☐ 3 ☒ 4 ☐ 5

3. Which one of the following values of x satisfies that $(5, x)$ is in the composed relation $S \circ R$?

   ☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5

4. Let R^* denote the transitive closure of S. How many pairs (a, b) are there in R^* ?

   ☐ 6 ☐ 9 ☐ 10 ⊡ 12 ☐ 20 ⊡ 25

   Note: There is a misprint in this question. R^* is the transitive closure of R, not of S.
   The transitive closure of R has 12 pairs.
   The transitive closure of S has 25 pairs.
   12 and 25 will both be considered to be correct answer.

Problem 8 (7 %)

A graph G with vertices $v_1, v_2, \ldots, v_7$ has adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

1. Answer the following true/false problems.

G has an Euler circuit	☒ True	☐ False
G has a Hamilton circuit	☐ True	☒ False
G has a Hamilton path	☐ True	☒ False

2. What is the length of the shortest simple circuit in G?

☐ 1 ☐ 2 ☐ 3 ☒ 4 ☐ 5 ☐ 6 ☐ 7

3. What is the number of edges of a spanning tree of G ?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☒ 6 ☐ 7 ☐ 8 ☐ 9

Problem 9 (4 %)

Which one of the following propositions is equivalent to $\forall x(P(x) \wedge \neg Q(x))$?

☐ $\neg\forall x(P(x) \vee Q(x))$
☒ $\neg\exists x(\neg P(x) \vee Q(x))$
☐ $\neg\forall x(\neg P(x) \wedge Q(x))$
☐ $\exists x(\neg P(x) \wedge Q(x))$

Problem 10 (4 %)

Answer the following problems about numbers of permutations and combinations.

$\forall n \forall r(P(n,r) = P(n,n-r))$	☐ True	☒ False
$\forall n \forall r(C(n,r) = C(n,n-r))$	☒ True	☐ False
$\forall n \forall r(P(n,r) = C(n,r) \cdot r!)$	☒ True	☐ False
$\forall n \forall r(C(n+1,r+1) = C(n,r))$	☐ True	☒ False

Problem 11 (4 %)

Consider the following statements:

Statement 1: If it is Sunday then it is sunshine.

Statement 2: If it is not Sunday then it is not sunshine.

Statement 3: If it is sunshine then it is Sunday.

Statement 4: If it is not sunshine then it is not Sunday.

What is the contrapositive of Statement 1?

☐ Statement 1 ☐ Statement 2 ☐ Statement 3 ☒ Statement 4

Problem 12 (6 %)

Let $A = \{\emptyset, 2, \{1,2\}\}$ and $B = \{a, \{\emptyset\}, 1, 2, 3\}$ be sets.

1. What is the cardinality of $A \cap B$?

   ☐ 0 ☒ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8

2. What is the cardinality of $A \cup B$?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☒ 7 ☐ 8

3. Which one of the following is an element of $A \times B$?

   ☒ $(2,1)$ ☐ $(1,2)$ ☐ $\{1,2\}$ ☐ $(\emptyset, \emptyset)$

4. Which one of the following is an element of the power set $\mathcal{P}(A)$?

   ☐ $\{\{\emptyset\}\}$ ☐ $\{1,2\}$ ☒ $\{\{1,2\}\}$ ☐ $\{\emptyset, 1\}$

Problem 13 (4 %)

1. What is the value of $P(6,3)$?

   ☐ 6
   ☐ 30
   ☒ 120
   ☐ 720

2. What is the value of $C(7,3)$?

   ☐ 15
   ☐ 20
   ☒ 35
   ☐ 210

Problem 14 (6 %)

Let $f(x) = (2x+3)(x^3 - 5x + 1)$.
Answer the following true/false problems.

1. $f(x)$ is $O(x^3)$ ☐ True ☒ False
2. $f(x)$ is $O(x^4)$ ☒ True ☐ False
3. $f(x)$ is $O(x^5)$ ☒ True ☐ False
4. $f(x)$ is $\Omega(x^3)$ ☒ True ☐ False
5. $f(x)$ is $\Omega(x^4)$ ☒ True ☐ False
6. $f(x)$ is $\Omega(x^5)$ ☐ True ☒ False
7. $f(x)$ is $\Theta(x^3)$ ☐ True ☒ False
8. $f(x)$ is $\Theta(x^4)$ ☒ True ☐ False
9. $f(x)$ is $\Theta(x^5)$ ☐ True ☒ False

Problem 15 (10 %)

In this problem we use Dijkstra's algorithm (see Figure 2) on the graph in Figure 1.

1. What is the length of the shortest path from a to z (found by Dijkstra's algorithm)?

   ☐ 6 ☐ 7 ☒ 8 ☐ 9 ☐ 14

2. In what order are vertices added to the set S ?

   ☐ a, v_1, v_2, v_5, v_8
   ☐ a, v_1, v_4, v_5, v_8
   ☒ $a, v_1, v_3, v_2, v_4, v_6, v_7, v_5, v_8$
   ☐ $a, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$

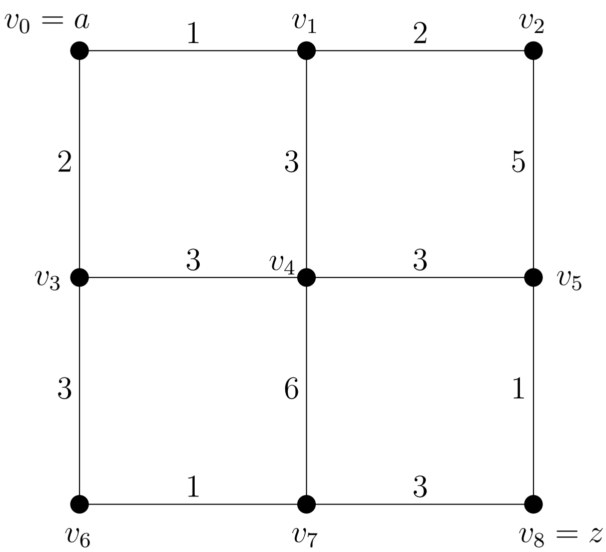

Figure 1: The graph G, considered in problems 15 and 16.

Problem 16 (5 %)

What is the weight of a minimum spanning tree of the graph in Figure 1.

☐ 9 ☐ 11 ☐ 12 ☐ 13 ☐ 14 ☐ 15 ☒ 16 ☐ 18

procedure *Dijkstra*(G: weighted connected simple graph, with
    all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
    where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
    $L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
    other labels are ∞, and S is the empty set}
while $z \notin S$
    $u :=$ a vertex not in S with $L(u)$ minimal
    $S := S \cup \{u\}$
    for all vertices v not in S
        if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
        {this adds a vertex to S with minimal label and updates the
        labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

Figure 2: