Answers to Exam in Discrete Mathematics

First Year at The TEK-NAT Faculty

June 15th, 2015, 9.00-13.00

Part I: (”Regular exercises”)

Exercise 1 (8%)

A sequence $a_0, a_1, a_2, a_3, \dots$ of integers is defined recursively by

$a_0 = 1,$
$a_{n+1} = 2a_n - n,$ for all $n \geq 0$.

Prove by induction that $a_n = n + 1$ for all $n \geq 0$.

Answer:
Basis step: Prove that the statement is true for $n = 0$.
Inductive step: Let $k \geq 0$ and assume that $a_k = k + 1$.
Then $a_{k+1} = 2a_k - k = 2(k+1) - k = (k+1) + 1$, using first the recursive definition of a_{k+1} and then the inductive hypothesis.
It follows that if the statement is true for $n = k$ then it is also true for $n = k + 1$.
And then by mathematical induction, $a_n = n + 1$ for all $n \geq 0$.

Exercise 2 (9%)

Consider the following algorithm.

procedure *sum*(n: positive integer)
$i := 1$
$x := 1$
$s := 1$
while $i < n$
  $i := i + 1$
  $x := x + 2$
  $s := s + x$
return s

1. Prove that the following assertion is a loop invariant for the while-loop:

$$i \in \mathbb{N} \;\wedge\; i \leq n \;\wedge\; x = 2i - 1 \;\wedge\; s = i^2. \tag{1}$$

2. What is the value of s in terms of n when the algorithm terminates? Justify your answer.

Answer:
Suppose that (1) is true and the condition $i < n$ is true before some iteration of the while loop. Then
$i_{\text{new}} = i + 1 \in \mathbb{N}$ and $i_{\text{new}} \leq n$,
$x_{\text{new}} = x + 2 = 2i - 1 + 2 = 2i + 1 = 2(i + 1) - 1 = 2i_{\text{new}} - 1$, and
$s_{\text{new}} = s + x_{\text{new}} = i^2 + 2i + 1 = (i + 1)^2 = i^2_{\text{new}}$.
We see that (1) is also true after the iteration of the while loop, and so (1) is a loop invariant.

Before the first iteration of the while loop, we have
$i = 1 \in \mathbb{N}$,
$i = 1 \leq n$, as n is a *positive* integer,
$x = 1 = 2i - 1$, and
$s = 1 = i^2$.
We see that (1) is true before while. Since (1) is a loop invariant, it is also true after the last iteration of while.

We the algorithm terminates we have:
The condition $i < n$ is false, otherwise the loop would continue.
$i \leq n$ is true, as it is part of (1).
Therefore $i = n$. We also have from (1) that $s = i^2$.
It follows that $s = n^2$ when the algorithm terminates.

Part II: (”Multiple choice” exercises)

There is only one correct answer to each question.

Exercise 3 (3%)

Using the extended Euclidean algorithm we find that

$$\gcd(258, 369) = -10 \cdot 258 + 7 \cdot 369 = 3.$$

Which one of the following statements is true?

☐ -10 is an inverse of 258 modulo 369.

☐ is an inverse of 258 modulo 369.

☐ 7 is an inverse of 258 modulo 369.

☒ 258 has no inverse modulo 369.

Exercise 4 (4%)

Which one of the following sets is *not* countable?

☐ The set of prime numbers
☒ $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$
☐ $\mathbb{Z} \times \mathbb{Z}$
☐ $\{x \in \mathbb{Q} \mid -3 \leq x \leq \sqrt{2}\}$

Exercise 5 (4%)

Which one of the following propositions is equivalent to $\forall x \exists y P(x, y)$?

☐ $\forall y \exists x P(x, y)$ ☐ $\exists x \forall y P(x, y)$ ☒ $\forall y \exists x P(y, x)$ ☐ $\exists y \forall x P(x, y)$

Exercise 6 (8%)

Consider the Merge Sort algorithm on page 360 in [Rosen, Discrete Mathematics and its Applications, Seventh Edition, Global Edition] using procedure *merge* on page 361.
Let $P(x, y)$ denote the statement: "if we use Merge Sort and procedure merge to sort the list

$$5, 2, 7, 3, 6, 1, 9, 4$$

then at some step we will directly compare x and y."
What is the truth value of each of the following propositions:

a. $P(3, 6)$

☐ True ☒ False

b. $P(3, 4)$

☒ True ☐ False

c. $P(2, 7)$

☐ True ☒ False

d. $P(1, 4)$

☒ True ☐ False

Exercise 7 (5%)

a. The propositions $r \rightarrow s$ and $\neg r \vee s$ are equivalent.

☒ True ☐ False

b. How many rows appear in a truth table of the compound proposition

$$p \vee \neg q \leftrightarrow \neg p \vee q$$

☐ 1 ☐ 2 ☐ 3 ☒ 4 ☐ 6 ☐ 8

c. In how many rows in this truth table is the truth value of $p \vee \neg q \leftrightarrow \neg p \vee q$ "true (T)" ?

☐ 0 ☐ 1 ☒ 2 ☐ 3 ☐ 4 ☐ 5

d. $p \vee \neg q \leftrightarrow \neg p \vee q$ is a tautology.

☐ True ☒ False

Exercise 8 (3%)

Which rule of inference is used in the following argument:

"If it is Valdemar's day then there are flags on the buses. It is Valdemar's day. Therefore, there are flags on the buses."

☐ Conjunction
☐ Modus tollens
☒ Modus ponens
☐ Universal generalization

Exercise 9 (4%)

Consider the following set of integers

$$S = \{x \mid 0 \leq x < 280 \ \wedge \ x \equiv 3 \pmod 7 \ \wedge \ x \equiv 4 \pmod 8\}.$$

How many integers are there in S?

☐ 0 ☐ 1 ☐ 2 ☒ 5 ☐ 10 ☐ 280

Exercise 10 (5%)

Let $(x-y)^5 = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5$, where a, b, c, d, e, f are integers.

a. One has that $b = e$

☐ YES ☒ NO

b. One has that d is equal to

☐ 5 ☐ 10 ☐ 20 ☐ −5 ☒ −10 ☐ −20

Let $(3x+2y)^3 = gx^3 + hx^2y + ixy^2 + jy^3$, where g, h, i, j are integers.

c. One has that h is equal to

☐ 6 ☐ 18 ☐ 27 ☐ 36 ☒ 54 ☐ 81

Exercise 11 (7%)

Let $A = \{1, 2, 3, 4, 5\}$ be a set. Consider the following two relations on A

$S = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)\}$

$$R = \{(2, 1), (2, 3), (4, 1), (4, 3), (4, 4), (4, 5), (5, 1)\}$$

Answer the following true/false exercises:

a. R is transitive

☒ True ☐ False

b. R is reflexive

☐ True ☒ False

c. S is an equivalence relation

☐ True ☒ False

d. $(1, 3)$ is in the transitive closure of S

☒ True ☐ False

e. $(2, 5)$ is in the transitive closure of S

☐ True ☒ False

f. $(3, 5)$ is in the composed relation $R \circ S$

☒ True ☐ False

g. $(3, 5)$ is in the composed relation $S \circ R$

☐ True ☒ False

Exercise 12 (7%)

$(111 \cdot 11113 + 1111115) \mod 11$ is equal to

☐ 0	☐ 1	☐ 2	☐ 3	☐ 4	☐ 5
☐ 6	☐ 7	☒ 8	☐ 9	☐ 10	

Exercise 13 (9%)

Let $f(x) = (x \log x + 5x)(x^2 + 3x - 4)$, for $x > 0$.
Answer the following 6 true/false exercises.

a. $f(x)$ is $O(x^3)$

☐ True ☒ False

b. $f(x)$ is $O(x^4)$

☒ True ☐ False

c. $f(x)$ is $O(x^3 \log x)$

☒ True ☐ False

d. $f(x)$ is $\Theta(x^3 \log x)$

☒ True ☐ False

e. $f(x)$ is $\Omega(x^3)$

☒ True ☐ False

f. $f(x)$ is $O(x^2 \log x)$

☐ True ☒ False

Exercise 14 (6%)

Let $f(x) = 3x^3 + 2x + 4$. One has that $f(x)$ is $O(x^3)$.

a. $(C, k) = (10, 0)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☐ True ☒ False

b. $(C, k) = (6, 1)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☐ True ☒ False

c. $(C, k) = (9, 1)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☒ True ☐ False

d. $(C, k) = (12, 1)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

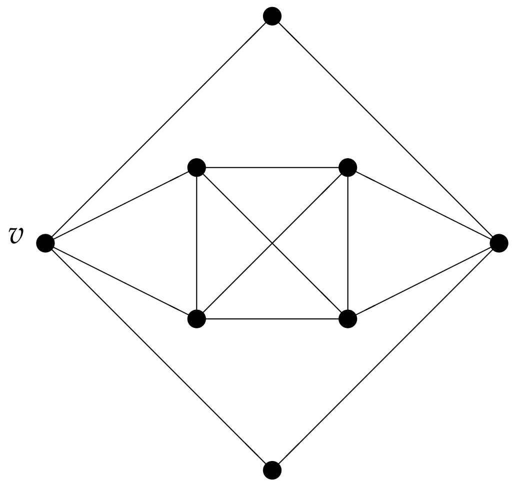
☒ True ☐ False

e. $(C, k) = (3, 2)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☐ True ☒ False

f. $(C, k) = (5, 2)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☒ True ☐ False



Figur 1: The graph G considered in Exercises 16 and 17.

Exercise 15 (6%)

Let $A = \{\emptyset, 1, 2, 3, 4\}$ and $B = \{\{\emptyset\}, 2, 4, 6\}$ be sets.

1. What is the cardinality of $A \cap B$?

   ☒ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8

2. What is the cardinality of $A \cup B$?

   ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☒ 7 ☐ 8

3. What is the cardinality of $A \times B$?

   ☐ 12 ☐ 15 ☐ 16 ☒ 20 ☐ 25 ☐ 30

4. Which one of the following is an element of $A \times B$?

   ☐ $\{\emptyset, \emptyset\}$ ☐ $(\emptyset, \emptyset)$ ☒ $(\emptyset, \{\emptyset\})$ ☐ $(\{\emptyset\}, 6)$

Exercise 16 (6%)

Consider the graph G in Figure 1.
Answer the following true/false questions.

a. G is a simple graph.

☒ True ☐ False

b. G is connected.

☒ True ☐ False

c. G has an Euler circuit.

☒ True ☐ False

d. G has a Hamilton circuit.

☐ True ☒ False

e. G has a Hamilton path.

☒ True ☐ False

Exercise 17 (6%)

Consider again the graph G in Figure 1.

a. What is degree of the vertex v

☐ 1 ☐ 2 ☐ 3 ☒ 4 ☐ 5 ☐ 6

b. What is the largest number of vertices in a complete subgraph of G

☐ 1 ☐ 2 ☐ 3 ☒ 4 ☐ 5 ☐ 6

c. What is the length of a shortest *simple* circuit of G

☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5 ☐ 6

d. What is the number of edges in a spanning tree of G

☐ 0 ☐ 1 ☐ 6 ☒ 7 ☐ 8 ☐ 14