Reexam in Discrete Mathematics

First Year at the Faculty of Engineering and Science and the Technical Faculty of IT and Design

August 20th, 2019, 9.00-13.00

This exam consists of 11 numbered pages with 12 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: ____________________________________

STUDENT NUMBER: ____________________________________

There is only one correct answer to each question.

Problem 1 (8 %)

Let $\mathcal{R}$ be a relation on a set S. If $\mathcal{R} = \emptyset$ (empty set) then
What are the truth values of the following statements?

If $S \neq \emptyset$ then

1. $\mathcal{R}$ is reflexive. ☐ True ☐ False
2. $\mathcal{R}$ is symmetric. ☐ True ☐ False

If $S = \emptyset$ then

3. $\mathcal{R}$ is reflexive. ☐ True ☐ False
4. $\mathcal{R}$ is symmetric. ☐ True ☐ False

Problem 2 (8 %)

Determine whether each of the following statements is true or false.

1. The set of real numbers between 0 and 5 is countable.

   ☐ TRUE ☐ FALSE

2. The set $A \times \mathbb{Z}^+$, where $A = \{1, 2\}$ and $\mathbb{Z}^+$ the set of positive integers, is countable.

   ☐ TRUE ☐ FALSE

3. If A and B are sets, A is uncountable and $A \subseteq B$ then B is uncountable.

   ☐ TRUE ☐ FALSE

4. If A is an infinite set, then it contains a countably infinite subset.

   ☐ TRUE ☐ FALSE

Problem 3 (8 %)

1. Which one of the following integers x satisfies that

$$\begin{aligned} x &\equiv 1 \ (\mathbf{mod}\ 3) \\ x &\equiv 1 \ (\mathbf{mod}\ 4) \\ x &\equiv 1 \ (\mathbf{mod}\ 5) \\ x &\equiv 0 \ (\mathbf{mod}\ 7) \end{aligned}$$

☐ 151 ☐ 156 ☐ 305 ☐ 457 ☐ 16 ☐ 301 ☐ 101

2. For which value of x is $17x \equiv 1 \ (\mathbf{mod}\ 43)$?

☐ 42 ☐ 47 ☐ 82 ☐ 38 ☐ 39 ☐ 10 ☐ 33

Problem 4 (8 %)

Determine whether each of the following theorems is true or false.

1. $(\mathbb{Z}, \neq)$ is a poset.

☐ TRUE ☐ FALSE

2. $(\mathbb{Z}, \geq)$ is a poset.

☐ TRUE ☐ FALSE

Problem 5 (9 %)

Answer the following true/false problems.

1. $\frac{x^3+2x}{2x+1}$ is $O(x^2)$ ☐ True ☐ False
2. $n\log(n^2+1)+n^2\log n$ is $\Omega(n^2\log n)$ ☐ True ☐ False
3. $\log_{10}x$ is $\Theta(\log_2 x)$ ☐ True ☐ False

Problem 6 (8 %)

1. What is the value of $50!$ **mod** 49?

   ☐ 50 ☐ 48 ☐ 1 ☐ 0 ☐ 3 ☐ 46 ☐ 49

2. What is the value of 2^{50} **mod** 17?

   ☐ 4 ☐ 1 ☐ 3 ☐ 5 ☐ 6 ☐ 16 ☐ 13

Problem 7 (8 %)

Consider the following algorithm:

procedure *smallest* ($a_1, a_2, \ldots, a_n$: natural numbers)
  $min := a_1$
  for $i := 2$ **to** n
    if $a_i < min$ **then** $min := a_i$
  return min {the smallest integer among the input}

What is the number of comparisons used by this algorithm to find the smallest natural number in a sequence of n natural numbers?

☐ $O(2n-1)$ ☐ $O(\frac{1}{3^{2n}})$ ☐ $O(2)$ ☐ $O(\frac{1}{3n})$ ☐ $O(\sqrt{n})$

Problem 8 (9 %)

1. A string that contains only 0s, 1s and 2s is called a **ternary string.**

   Find a recurrence relation for the number of ternary strings of length n that contain two consecutive 0s.

   ☐ $a_n = 2a_{n-1} + 3^{n-1}$ for all $n \geq 1$.
   ☐ $a_n = 2a_{n-1} + 4a_{n-2}$ for all $n \geq 2$.
   ☐ $a_n = 3a_{n-1} + 2a_{n-2} + 2^n$ for all $n \geq 3$.
   ☐ $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$ for all $n \geq 2$.

2. What are the initial conditions?

   ☐ $a_0 = 0,\ a_1 = 2$
   ☐ $a_0 = 0,\ a_1 = 1$
   ☐ $a_0 = 0,\ a_1 = 0$
   ☐ $a_0 = 1,\ a_1 = 0$

3. How many ternary strings of length four do not contain two consecutive 0s?

   ☐ 11 ☐ 8 ☐ 17 ☐ 281 ☐ 79 ☐ 19 ☐ 21

Problem 9 (8 %)

1. What is the coefficient of x^k in the expansion of $(x+\frac{1}{x})^{100}$, where k is an integer?

- ☐ $P(100, \frac{100-k}{2})$.
- ☐ $2^{100}P(100, k)$
- ☐ $C(100, k)$
- ☐ 2^{100}
- ☐ $C(100, \frac{100-k}{2})$

2. Which expression below is equal to the identity

$$C(n, r) \cdot C(r, k)$$

whenever n, r and k are nonnegative integers with $r \leq n$ and $k \leq r$?

- ☐ $C(n-k, k) \cdot C(n-k, r)$
- ☐ $C(n-k, k) \cdot C(n-k, k)$
- ☐ $C(n, k) \cdot C(n-k, r-k)$
- ☐ $C(n, k) \cdot C(n, r)$

Problem 10 (8 %)

Theorem Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

Which one (s) of the followings is (are) wrong?

a) The statement $P(2)$ is: $2! < 2^2$.

b) The basis step of the proof is proving that $P(2)$ is true.

c) The inductive hypothesis is the statement that $k! < k^k$.

d) In the inductive step, we prove that the statement that $k! < k^k$ is true.

e) In the inductive step, we want to show for each $k \geq 2$ that $P(k)$ implies $P(k+1)$.

f) The statement e) is the same as showing that assuming the inductice hypothesis, one can prove that $(k+1)! < (k+1)^{k+1}$.

☐ c, d ☐ d, e, f ☐ c, d, e ☐ d, e ☐ d

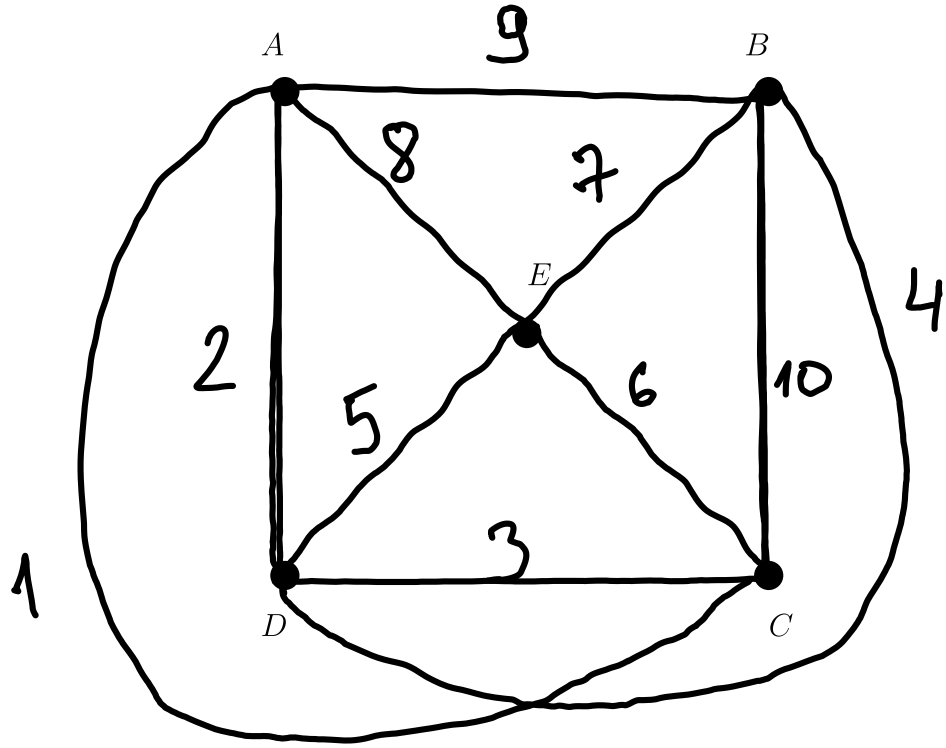

Figure 1: The graph G, considered in Problem 11.

Problem 11 (9 %)

In this problem we use Dijkstra's algorithm on the graph G in Figure 1.

1. What is the length of the shortest path from A to E (found by Dijkstra's algorithm)?

   ☐ 8 ☐ 5 ☐ 7 ☐ 11

2. In what order are vertices added to the set S in Dijkstra's algorithm (see Figure 2)?

   ☐ A, C, B, D, E
   ☐ A, E, C, D, E
   ☐ A, B, C, D, E
   ☐ A, C, D, B, E

3. What is the weight of a minimum spanning tree of the graph G in Figure 1.

   ☐ 12
   ☐ 17
   ☐ 14
   ☐ 11

Problem 12 (9%)

1. Consider the possible directed graphs G with 4 nodes. Is it possible for G to have 23 edges?

   ☐ TRUE ☐ FALSE

2. Let G be a graph with no loops or multipy edges. If G has 7 vertices and 5 edges then the graph G is definitely **not** a tree.

   ☐ TRUE ☐ FALSE

3. Let G be a graph with no loops or multipy edges. If G has 7 vertices and an Euler circuit then the graph G is definitely a tree.

   ☐ TRUE ☐ FALSE

procedure *Dijkstra*(G: weighted connected simple graph, with
all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
$L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
other labels are ∞, and S is the empty set}
while $z \notin S$
$u :=$ a vertex not in S with $L(u)$ minimal
$S := S \cup \{u\}$
for all vertices v not in S
if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
{this adds a vertex to S with minimal label and updates the
labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

Figure 2: