Reexam in Discrete Mathematics

First Year at the Faculty of Engineering and Science and the Technical Faculty of IT and Design

August 14th, 2018, 9.00-13.00

This exam consists of 11 numbered pages with 14 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: ____________________________________

STUDENT NUMBER: ____________________________________

There is only one correct answer to each question.

Problem 1 (6 %)

Which one is the contrapositive of the theorem "if $2^n - 1$ is a prime number then n is prime" for $n \in \mathbb{N}$?

- ☐ if $2^n - 1$ is a prime number then n is prime.
- ☐ if n is prime then $2^n - 1$ is not a prime number.
- ☐ if n is prime then $2^n - 1$ is a prime number.
- ☐ if n is not prime then $2^n - 1$ is not a prime number.
- ☐ if $2^n - 1$ is not a prime number then n is not prime.

Problem 2 (8 %)

Suppose that the variable x represents students, y represents courses, and $T(x, y)$ means "x is taking y". Match the English statement with all its equivalent symbolic statements in this list:

1. Every course is being taken by at least one student.

   ☐ $\exists x\ \forall y\ T(x, y)$　　☐ $\forall x\ \forall y\ T(x, y)$　　☐ $\forall y\ \exists x\ T(x, y)$

2. There is a course that no students are taking.

   ☐ $\exists x\ \forall y\ T(x, y)$　　☐ $\forall x\ \forall y\ T(x, y)$　　☐ $\exists y\ \forall x\ \neg T(x, y)$

3. No student is taking any course.

   ☐ $\neg\exists x\ \exists y\ T(x, y)$　　☐ $\exists x\ \forall y\ T(x, y)$　　☐ $\exists x\ \forall y\ T(y, x)$

4. There is a course that all students are taking.

   ☐ $\exists y\ \forall x\ T(x, y)$　　☐ $\exists x\ \forall y\ T(x, y)$　　☐ $\forall x\ \forall y\ T(x, y)$

Problem 3 (6 %)

Find the solution of the linear congruence $15x \equiv 31 \pmod{47}$ given that the inverse of 15 modulo 47 is 22.

☐ 10 ☐ 29 ☐ 2 ☐ 7 ☐ 24 ☐ 1 ☐ 23

Problem 4 (8 %)

Answer the following true/false problems.

1. $(x+2)\log(x^2+1)$ is $O(x \log x)$ ☐ True ☐ False
2. $3x^4 + \log x^8$ is $O(x^3)$ ☐ True ☐ False
3. $x \log x$ is $\Omega(\log(\log x))$ ☐ True ☐ False
4. $\log(x^2+1)$ is $\Omega(\log 2^x)$ ☐ True ☐ False

Problem 5 (8 %)

1. What is the value of $(12^2 \textbf{ mod } 17)^3 \textbf{ mod } 11$?

   ☐ 7 ☐ 6 ☐ 1 ☐ 4 ☐ 3

2. What is $lcm(2^{89}, 2^{346})$?

   ☐ 2^{89} ☐ 3^9 ☐ 4 ☐ 2^{346} ☐ 6

Problem 6 (6 %)

Determine the type and the order of the given recurrence relation

$$a_{n+3} = 2a_{n+2} + a_{n+1} - a_n, \ n \geq 0$$

1. It is linear

   ☐ True ☐ False

2. It is homogeneous

   ☐ True ☐ False

3. It is of the 2nd order

   ☐ True ☐ False

Problem 7 (6 %)

Find the coefficient of x^5 in $(2 - x^2)^{12}$

☐ 12 ☐ -2 ☐ 0 ☐ 2 ☐ -15

Problem 8 (8 %)

Find the number of squares denoted by S_n of **all sizes** in an $n \times n$ (n is a positive integer) square.

☐ $S_n = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$

☐ $S_n = 5n^2 - n + 1$ for $n \geq 2$

☐ $S_n = n^3 + 2n - 1$ for $n \geq 4$

☐ $S_n = \frac{n(n^2+3n+2)}{4}$ for $n \geq 0$

☐ $S_n = 1$ for $n \geq 1$

Problem 9 (8 %)

Let $P(n)$ be the following statement

$$\sum_{k=1}^{n} \frac{1}{k \cdot (k+1)} = \frac{n}{n+1}$$

We want to prove by induction that $P(n)$ is true for all $n \geq 1$.

1. What is the correct basis step of the induction proof?
   - ☐ Prove that $P(1)$ is true
   - ☐ Prove that $P(2)$ is true
   - ☐ Prove that $P(0)$ is true
   - ☐ Prove that $P(n)$ is true, for all $n \geq 1$

2. Which one of the following is a correct outline of the inductive step?
   - ☐ Let $i \geq -1$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=1}^{i} \frac{1}{k \cdot (k+1)} = \frac{i}{i+1}$. Use this to prove $P(i)$.
   - ☐ Let $i \geq 0$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=1}^{i} \frac{1}{k \cdot (k+1)} = \frac{i}{i+1}$. Use this to prove $P(i+1)$.
   - ☐ Let $i = 0$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=1}^{i} \frac{1}{k \cdot (k+1)} = \frac{i}{i+1}$. Use this to prove $P(i)$.
   - ☐ Let $i \geq 0$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=1}^{i} \frac{1}{k \cdot (k+1)} = \frac{i}{i+1}$. Use this to prove $P(i+2)$.
   - ☐ Let $i \geq 1$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=1}^{i} \frac{1}{k \cdot (k+1)} = \frac{i}{i+1}$. Use this to prove $P(i+1)$.

Problem 10 (9 %)

Let $A = \{1, 2, 4, 5, 7, 11, 13\}$ be given. Let a relation $\mathcal{R}$ on A be defined by $(x, y) \in \mathcal{R}$ if and only if $x - y$ is a multiple of 3. That is

$$\mathcal{R} = \{(i, j) \mid 3 \text{ divides } (i - j)\}.$$

1. Answer the following true/false problems.

$\mathcal{R}$ is reflexive	☐ True	☐ False
$\mathcal{R}$ is symmetric	☐ True	☐ False
$\mathcal{R}$ is transitive	☐ True	☐ False
$\mathcal{R}$ is an equivalence relation	☐ True	☐ False

2. What is the equivalence class of [1]?

☐ $\{1, 4, 7, 13\}$ ☐ $\{0, 3, 6, 12\}$ ☐ $\{1, 4, 8, 11\}$

3. What is the partition of A with respect to $\mathcal{R}$ (if it exits)

☐ $[1] \bigcup [2]$ ☐ $[0] \bigcup [2]$ ☐ does not exit.

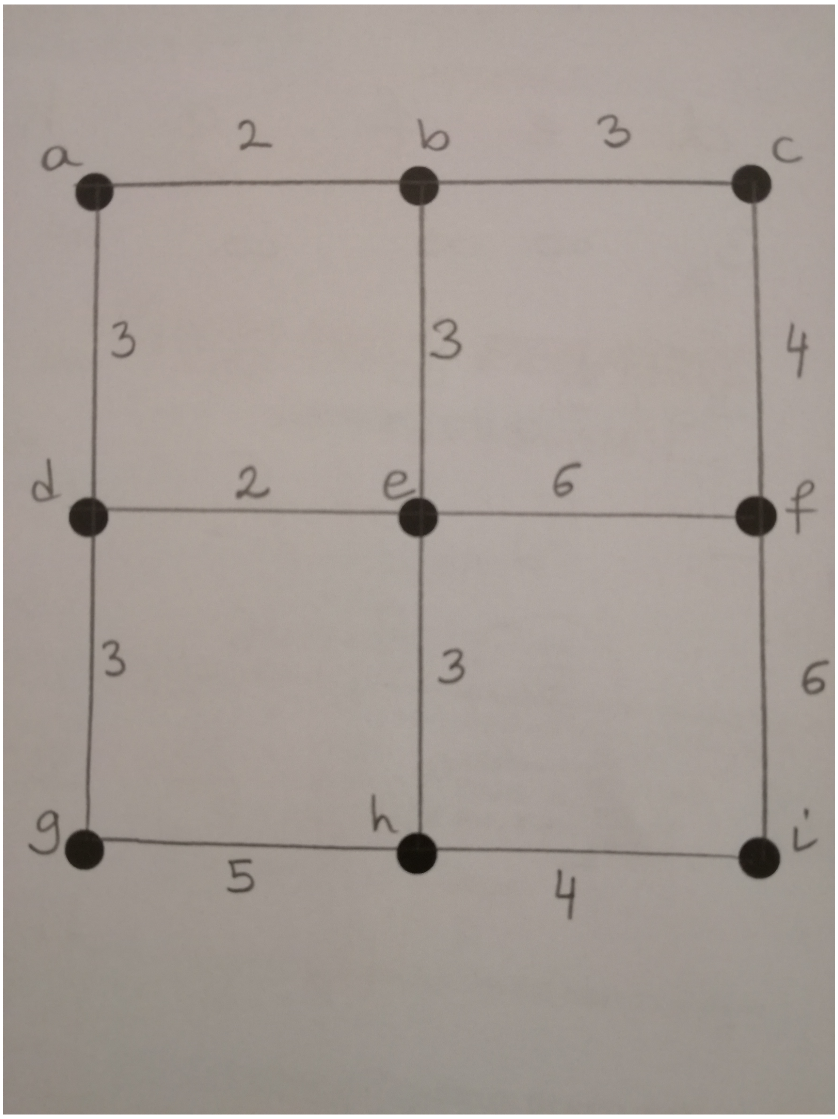

Figure 1: The graph G, considered in Problems 11, 12 and 13.

Problem 11 (10 %)

In this problem we use Dijkstra's algorithm (see Figure 2 on page 11) on the graph G in Figure 1.

1. What is the length of the shortest path from a to i (found by Dijkstra's algorithm)?

   ☐ 17 ☐ 18 ☐ 15 ☐ 11 ☐ 12 ☐ 24

2. In what order are vertices added to the set S ?

   ☐ a, c, d, e, g
   ☐ $a, b, d, c, e, g, h, f, i$
   ☐ $a, b, c, d, e, f, g, h, i$
   ☐ a, e, b, f, g
   ☐ a, d, g, h, i
   ☐ a, b, c, f, i

Problem 12 (5 %)

What is the weight of a minimum spanning tree of the graph G in Figure 1.

☐ 15 ☐ 16 ☐ 22 ☐ 24 ☐ 12 ☐ 13

Problem 13 (6 %)

Answer the following questions.

1. If a connected graph has only two odd vertices, does it have any Euler path

   ☐ Yes ☐ No

2. Does the complete graph with 5 vertices has an Euler path?

   ☐ Yes ☐ No

Problem 14 (6 %)

Answer the following questions.

1. How many edges does the complete graph with N vertices have?

   ☐ N-1 ☐ N-2 ☐ N^2 ☐ $2N$ ☐ $\frac{N(N-1)}{2}$

2. What is the degree of every vertex in a complete graph with N vertices?

   ☐ N-1 ☐ N-2 ☐ N^2 ☐ $2N$ ☐ $\frac{N(N-1)}{2}$

3. How many edges does a tree with 10 vertices have?

   ☐ 10 ☐ 5 ☐ 9 ☐ 20 ☐ 45

procedure *Dijkstra*(G: weighted connected simple graph, with
    all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
    where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
    $L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
    other labels are ∞, and S is the empty set}
while $z \notin S$
    $u :=$ a vertex not in S with $L(u)$ minimal
    $S := S \cup \{u\}$
    for all vertices v not in S
        if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
        {this adds a vertex to S with minimal label and updates the
        labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

Figure 2: