Reexam in Discrete Mathematics

First Year at The TEK-NAT Faculty

August 23, 2016, 9.00-13.00

This exam consists of 11 numbered pages with 16 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: ____________________________________

STUDENT NUMBER: ____________________________________

There is only one correct answer to each question.

Problem 1 (8 %)

Let $f(x) = 2x^3 + 3x^2 \log x + 7x + 1$, for $x > 0$.
Answer the following true/false problems.

1. $f(x)$ is $O(x^3 \log x)$ ☒ True ☐ False
2. $f(x)$ is $O(x^3)$ ☒ True ☐ False
3. $f(x)$ is $O(x^2 \log x)$ ☐ True ☒ False
4. $f(x)$ is $\Omega(x^3 \log x)$ ☐ True ☒ False
5. $f(x)$ is $\Omega(x^3)$ ☒ True ☐ False
6. $f(x)$ is $\Omega(x^2 \log x)$ ☒ True ☐ False
7. $f(x)$ is $\Theta(x^3 \log x)$ ☐ True ☒ False
8. $f(x)$ is $\Theta(x^3)$ ☒ True ☐ False
9. $f(x)$ is $\Theta(x^2 \log x)$ ☐ True ☒ False

Problem 2 (4 %)

Which one of the following numbers is an inverse of 17 modulo 50?

☐ -3 ☐ 1 ☒ 3 ☐ 11 ☐ 33

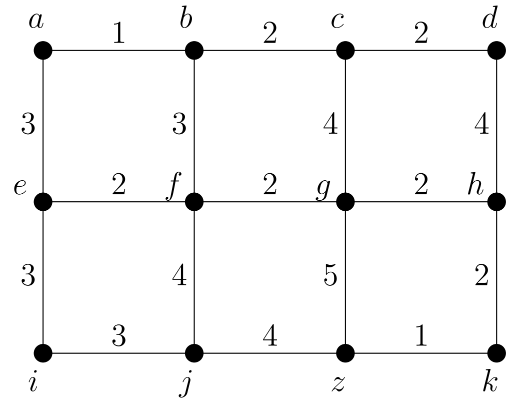

Figure 1: The graph G, considered in Problems 3, 4 and 5.

Problem 3 (8 %)

In this problem we use Dijkstra's algorithm (see Figure 2 on Page 11) on the graph in Figure 1.

1. What is the length of the shortest path from a to z (found by Dijkstra's algorithm)?

   ☐ 10 ☒ 11 ☐ 12 ☐ 13 ☐ 14

2. Which one of the following vertices is added first to the set S

   ☒ d ☐ g ☐ h ☐ i ☐ j

3. Which one of the following vertices is the last to be added to the set S

   ☐ h ☐ i ☐ j ☐ k ☒ z

Problem 4 (5 %)

What is the weight of a minimum spanning tree of the graph in Figure 1.

☐ 19 ☐ 20 ☐ 21 ☐ 22 ☒ 23 ☐ 24 ☐ 25 ☐ 26

Problem 5 (6 %)

In this problem G is the graph in Figure 1. (The edge weights of G are not considered in this problem.)

1. Answer the following true/false problems.

   G has an Euler circuit ☐ True ☒ False

   G has a Hamilton circuit ☒ True ☐ False

2. What is the number of edges of a spanning tree of G ?

   ☐ 1 ☐ 3 ☐ 5 ☐ 7 ☐ 9 ☒ 11 ☐ 13 ☐ 15 ☐ 17

3. What is the degree of the vertex z ?

   ☐ 0 ☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5 ☐ 11

Problem 6 (10 %)

A sequence of numbers $a_1, a_2, a_3, a_4, \ldots$ is defined recursively by

- $a_1 = 0$
- For $n \geq 2$ let m be an integer such that $n = 2m$ or $n = 2m + 1$. Then $a_n = a_m + 1$.

Recall that $\log x$ denotes the base 2 logarithm of x and that $\lfloor x \rfloor$ is the largest integer less than or equal to x. E.g. $a_n = a_{\lfloor \frac{n}{2} \rfloor} + 1$.

Let $P(n)$ denote the following assertion

$$a_n = \lfloor \log n \rfloor.$$

We want to prove by induction or strong induction that $P(n)$ is true for every integer $n \geq 1$.

1. What is the correct basis step of the induction proof

   ☐ Prove that $P(0)$ is true
   ☒ Prove that $P(1)$ is true
   ☐ Prove that $P(2)$ is true
   ☐ Prove that $P(n)$ is true, for all $n \leq 1$

2. Which one of the following is a correct outline of the inductive step?

   ☐ Let $k \geq 1$ and assume that $P(k)$ is true. Let $m = \lfloor \frac{k+1}{2} \rfloor$.
   By the induction hypothesis $2^{a_m} \leq m \leq 2^{a_m+1} - 1$. Use this to prove $P(k+1)$.
   ☐ Let $k \geq 1$ and assume that $P(k)$ is true. Let $m = \lfloor \frac{k+1}{2} \rfloor$.
   By the induction hypothesis $a_m \leq 2^m \leq a_m + 1$.
   Use this to prove $P(k+1)$.
   ☐ Let $k \geq 1$ and assume that $P(j)$ is true for all j where $0 \leq j \leq k$. Let $m = \lfloor \frac{k+1}{2} \rfloor$.
   By the induction hypothesis $2^{a_m} \leq m \leq 2^{a_m+1} - 1$. Use this to prove $P(k+1)$.
   ☒ Let $k \geq 1$ and assume that $P(j)$ is true for all j where $1 \leq j \leq k$. Let $m = \lfloor \frac{k+1}{2} \rfloor$.
   By the induction hypothesis $2^{a_m} \leq m \leq 2^{a_m+1} - 1$. Use this to prove $P(k+1)$.
   ☐ Let $k \geq 1$ and assume that $P(j)$ is true for all j where $1 \leq j \leq k$. Let $m = \lfloor \frac{k+1}{2} \rfloor$.
   By the induction hypothesis $a_m \leq 2^m \leq a_m + 1$.
   Use this to prove $P(k+1)$.

Problem 7 (6 %)

Consider the following linear homogeneous recurrence relation

$$a_n = 4a_{n-2}.$$

1. What is the degree of this recurrence relation?

   ☐ 0 ☐ 1 ☒ 2 ☐ 4

2. Which of the following is the solution of this recurrence relation (α_1 and α_2 are constants)?

   ☐ $a_n = \alpha_1(-2)^n + \alpha_2 \cdot 3^n$
   ☒ $a_n = \alpha_1(-2)^n + \alpha_2 \cdot 2^n$
   ☐ $a_n = \alpha_1 + \alpha_2(-2)^n$
   ☐ $a_n = \alpha_1 \cdot 2^n + \alpha_2$

Problem 8 (5 %)

Consider the following algorithm:

procedure *multiplications*(n: positive integer)
$t := 1$
for $i := 1$ **to** n
  $j := 1$
  while $j \leq n$
    $j := 2 \cdot j$
    $t := t + 1$
return t

The number of multiplications used by this algorithm is

☐ $O(n)$ ☐ $\Theta(n)$ ☒ $O(n \log n)$ ☐ $\Omega(n^2)$

Problem 9 (8 %)

1. Is the compound proposition $p \wedge q \rightarrow p \vee q$ a tautology?

   ☒ Yes ☐ No

2. Are the propositions $p \wedge q$ and $p \vee q$ equivalent?

   ☐ Yes ☒ No

3. How many rows appear in a truth table of the compound proposition

$$p \vee q \rightarrow p \wedge q$$

   ☐ 1 ☐ 2 ☐ 3 ☒ 4 ☐ 6 ☐ 8 ☐ 10

Problem 10 (4 %)

Which rule of inference is used in the following argument:
"If it is summer then it is sunshine. It is summer. Therefore, it is sunshine."

☐ Conjunction
☒ Modus ponens
☐ Modus tollens
☐ Hypothetical syllogism
☐ Universal generalization

Problem 11 (5 %)

What is the value of $(123 + 1234 + 12345 \cdot 222) \mod 10$?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☒ 7 ☐ 8 ☐ 9

Problem 12 (8 %)

Consider the following two relations on the set $A = \{1, 2, 3, 4, 5\}$:

$$R = \{(1,2),(1,4),(2,3),(3,1),(4,5),(5,1)\}$$

$$S = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}.$$

1. Answer the following true/false problems:

S is reflexive	☒ True	☐ False
S is antisymmetric	☒ True	☐ False
S is symmetric	☒ True	☐ False
S is transitive	☒ True	☐ False
R is transitive	☐ True	☒ False

2. Let R^* denote the transitive closure of R. How many pairs (a, b) are there in R^* ?

   ☐ 6 ☐ 9 ☐ 10 ☐ 12 ☐ 20 ☒ 25

Problem 13 (5 %)

Let $(x+2y)^6 = ax^6+bx^5y+cx^4y^2+dx^3y^3+ex^2y^4+fxy^5+gy^6$, where a, b, c, d, e, f, g are integers.

1. What is the value of c?

   ☐ 4 ☐ 15 ☒ 60 ☐ 64 ☐ 80

2. What is the value of g?

   ☐ 4 ☐ 15 ☐ 60 ☒ 64 ☐ 80

Problem 14 (6 %)

Let $A = \{1, 2, 3, \{1, 2\}\}$ and $B = \{\emptyset, \{\emptyset\}, \{1, 2\}, \{1, 3\}\}$ be sets.

1. What is the cardinality of $A \cap B$?

   ☐ 0 ☒ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8

2. What is the cardinality of $A \cup B$?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☒ 7 ☐ 8

3. What is the cardinality of $A \times B$?

   ☐ 9 ☐ 12 ☐ 15 ☒ 16 ☐ 20 ☐ 25

4. Which one of the following is an element of the power set $\mathcal{P}(B)$?

   ☒ $\{\{\emptyset\}\}$ ☐ $\{1\}$ ☐ $\{1, 2\}$ ☐ $\{\emptyset, 1, 2\}$

Problem 15 (4 %)

Which one of the following propositions is equivalent to $\forall x \exists y(\neg P(x) \wedge Q(y))$?

☐ $\neg \forall x \exists y(P(x) \wedge \neg Q(y))$

☒ $\neg \exists x \forall y(P(x) \vee \neg Q(y))$

☐ $\neg \exists y \forall x(P(x) \wedge \neg Q(y))$

☐ $\exists x \forall y(P(x) \vee \neg Q(y))$

☐ $\neg \exists y \forall x(P(x) \vee Q(y))$

Problem 16 (8 %)

Consider the following algorithm:

procedure *sequence*(n: positive integer)
$i := 0$
$x := 2$
while $i < n$
  $i := i + 1$
  $x := 3x + 2$
return x

1. Which one of the following statements is a loop invariant for the while loop in this algorithm?

   ☐ $i \leq n \quad \wedge \quad x = 3^i + 1$
   ☐ $i \leq n - 1 \quad \wedge \quad x = 3^i + 1$
   ☒ $i \leq n \quad \wedge \quad x = 3^{i+1} - 1$
   ☐ $i \leq n - 1 \quad \wedge \quad x = 3^i - 1$
   ☐ $i \leq n \quad \wedge \quad x = 3^n - 1$

2. What is the value of x returned by procedure *sequence*?

   ☐ $3^n + 1$ ☐ $3^{n+1} + 1$ ☐ $3^n - 1$ ☒ $3^{n+1} - 1$

procedure *Dijkstra*(G: weighted connected simple graph, with
all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
$L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
other labels are ∞, and S is the empty set}
while $z \notin S$
$u :=$ a vertex not in S with $L(u)$ minimal
$S := S \cup \{u\}$
for all vertices v not in S
if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
{this adds a vertex to S with minimal label and updates the
labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

Figure 2: