Exam in Discrete Mathematics

First Year at the Faculty of Engineering and Science and the Technical Faculty of IT and Design

June 4th, 2019, 9.00-13.00

This exam consists of 11 numbered pages with 12 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: ______________________________

STUDENT NUMBER: ______________________________

There is only one correct answer to each question.

Problem 1 (8 %)

Let $\mathcal{R}$ be a relation on the set of all integers and defined by $x \geq y^2$ and $\mathcal{S}$ be a relation on the set of all integers and defined by $x = y^2$.

What are the truth values of the following statements?

1. $\mathcal{R}$ is transitive. ☐ True ☐ False
2. $\mathcal{R}$ is antisymmetric. ☐ True ☐ False
3. $\mathcal{R}$ is an equivalence relation. ☐ True ☐ False
4. $\mathcal{S} \circ \mathcal{R} = \mathcal{R}$. ☐ True ☐ False

Problem 2 (8 %)

Determine whether each of the following statements is true or false.

1. The set of integers with the absolute value less than 1000 is countable.

   ☐ TRUE ☐ FALSE

2. If A is an uncountable set and B is a countable set then $A - B$ (the difference of A and B) must be uncountable.

   ☐ TRUE ☐ FALSE

3. It is true that $x^2 = y^2$ if and only if $x = y$ or $x = -y$.

   ☐ TRUE ☐ FALSE

4. If n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

   ☐ TRUE ☐ FALSE

Problem 3 (8 %)

1. Which one of the following integers x satisfies that

$$\begin{aligned} x &\equiv 1 \pmod{3} \\ x &\equiv 4 \pmod{5} \\ x &\equiv 6 \pmod{7} \end{aligned}$$

☐ 10 ☐ 34 ☐ 7 ☐ 32 ☐ 13 ☐ 24 ☐ 14

2. Find the value of x satisfying $27x \equiv 1 \pmod{392}$.

☐ 101 ☐ 347 ☐ 367 ☐ 363 ☐ 138 ☐ 242 ☐ 141

Problem 4 (8 %)

Determine whether each of the following theorems is true or false.

1. $(\mathbb{Z}, =)$ is a poset.

☐ TRUE ☐ FALSE

2. $(\mathbb{Z}, \nmid)$ is a poset.

☐ TRUE ☐ FALSE

Problem 5 (9 %)

Answer the following true/false problems.

1. $2^x + 17$ is $O(3^x)$ ☐ True ☐ False
2. $3x + 7$ is $\Omega(x)$ ☐ True ☐ False
3. $2x^2 + x - 7$ is $\Theta(x^2)$ ☐ True ☐ False

Problem 6 (8 %)

1. What is the value of 4^{532} **mod** 11?

   ☐ 2 ☐ 3 ☐ 1 ☐ 5 ☐ 4 ☐ 8 ☐ 7

2. What is 50! **mod** 50?

   ☐ 49 ☐ 3 ☐ 7 ☐ 51 ☐ 47 ☐ 46 ☐ 0

Problem 7 (8 %)

Consider the following algorithm:

procedure *count ones* ($a_1a_2 \dots a_n$: bit string)
count:=0 {no 1's yet}
for $i := 1$ **to** n
    if $a_i = 1$ then count:=count+1
return count {count contains the number of 1's}

Give a big-O estimate for the number of comparisons used by this algorithm.

☐ $O(\frac{\log n}{n})$ ☐ $O(\frac{1}{n})$ ☐ $O(1)$ ☐ $O(n)$ ☐ $O(\sqrt{n})$

Problem 8 (9 %)

1. Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

   ☐ $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$ for all $n \geq 3$.

   ☐ $a_n = a_{n-2} + a_{n-1} + 2^n$ for all $n \geq 2$.

   ☐ $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$ for all $n \geq 2$.

   ☐ $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 3$.

2. What are the initial conditions?

   ☐ $a_0 = a_1 = a_2 = 0$.

   ☐ $a_0 = 1$, $a_1 = 2$ og $a_2 = 0$.

   ☐ $a_0 = 0$, $a_1 = 2$ og $a_2 = 0$.

   ☐ $a_0 = 1$, $a_1 = 2$ og $a_2 = 2$.

3. How many bit strings of length five contain three consecutive 0s?

   ☐ 1 ☐ 5 ☐ 12 ☐ 8 ☐ 20 ☐ 3 ☐ 47

Problem 9 (8 %)

1. What is the coefficient of $x^{101}y^{99}$ in $(2x-3y)^{200}$?
   - ☐ $-2^{101}3^{99}C(200,99)$
   - ☐ 0
   - ☐ $-2^{101}3^{99}P(200,99)$
   - ☐ $-2^{101}3^{99}$
   - ☐ C(200,99)

2. Suppose that k, n are integers with $1 \leq k < n$. Which one is equal to

$$C(n-1,k-1)\cdot C(n,k+1)\cdot C(n+1,k)?$$

   - ☐ $C(n-1,k)\cdot C(n,k-1)\cdot C(n+1,k+1)$
   - ☐ $C(n-1,k-1)\cdot C(n+1,k+1)\cdot C(n,k)$
   - ☐ $C(n,k)\cdot C(n-1,k-1)\cdot C(n,k+1)$
   - ☐ $C(n-1,k-2)\cdot C(n+1,k+1)$

where $P(n,k)$ is the k-permutation of a set with n elements and where $C(n,k)$ is the k-combination of a set with n elements..

Problem 10 (8 %)

Theorem For every positive integer n, $\sum_{i=1}^{n} i = \frac{(n+\frac{1}{2})^2}{2}$.

Basis Step: The formula is true for $n = 1$.

Inductive Step: Suppose that $\sum_{i=1}^{n} i = \frac{(n+\frac{1}{2})^2}{2}$. Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1)$. By the inductive hypothesis, $\sum_{i=1}^{n+1} i = \frac{(n+\frac{1}{2})^2}{2} + n + 1 = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n + \frac{3}{2})^2/2 = [(n+1) + \frac{1}{2}]^2/2$, completing the inductive step.

What is wrong with this "proof"?

- ☐ Basis step must be verified for $n = 0$.
- ☐ The inductive step is wrong.
- ☐ Basis step is wrong.
- ☐ There is nothing wrong with this proof.

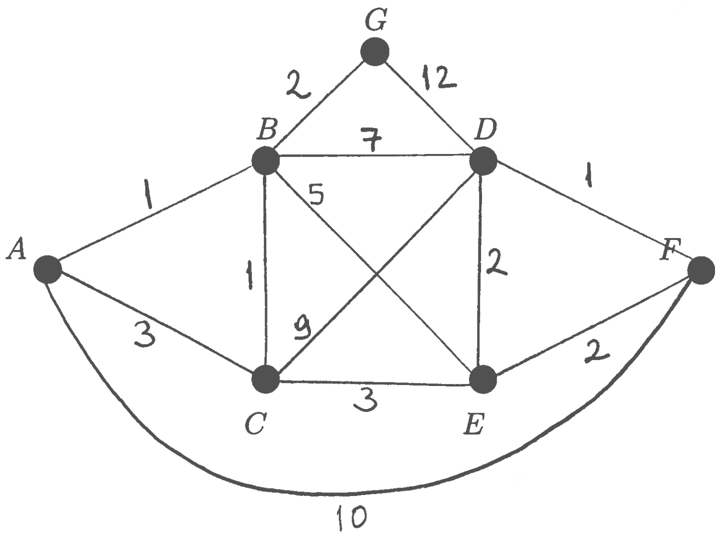

Figure 1: The graph G, considered in Problem 11.

Problem 11 (9 %)

In this problem we use Dijkstra's algorithm (see Figure 2) on the graph G in Figure 1.

1. What is the length of the shortest path from A to F?

   ☐ 10 ☐ 9 ☐ 8 ☐ 7

2. In what order are vertices added to the set S in Dijkstra's algorithm (see Figure 2)?

   ☐ A, C, G, B, D, E, F
   ☐ A, B, C, D, E, F
   ☐ A, B, C, G, E, D, F
   ☐ A, C, E, F

3. What is the weight of a minimum spanning tree of the graph G in Figure 1.

☐ 8
☐ 7
☐ 5
☐ 10

Problem 12 (9%)

Let K_n denote the complete, undirected graph with n nodes (i.e. every pair of the n-nodes contains an edge between them).

1. When (for what values of n) does K_n have an Euler path?

   ☐ When n is odd
   ☐ When n is even
   ☐ For all n
   ☐ None

2. When (for what values of n) does K_n have a Hamilton path?

   ☐ When n is even
   ☐ When n is odd
   ☐ For all n
   ☐ None

3. Is it possible to have a graph that has a Hamilton path but no Euler path?

   ☐ YES ☐ NO

```
procedure Dijkstra(G: weighted connected simple graph, with
    all weights positive)
{G has vertices a = v_0, v_1, ..., v_n = z and lengths w(v_i, v_j)
    where w(v_i, v_j) = ∞ if {v_i, v_j} is not an edge in G}
for i := 1 to n
    L(v_i) := ∞
L(a) := 0
S := ∅
{the labels are now initialized so that the label of a is 0 and all
    other labels are ∞, and S is the empty set}
while z ∉ S
    u := a vertex not in S with L(u) minimal
    S := S ∪ {u}
    for all vertices v not in S
        if L(u) + w(u, v) < L(v) then L(v) := L(u) + w(u, v)
        {this adds a vertex to S with minimal label and updates the
        labels of vertices not in S}
return L(z) {L(z) = length of a shortest path from a to z}
```

Figure 2: