Exam in Discrete Mathematics

First Year at the Faculty of Engineering and Science and the Technical Faculty of IT and Design

May 29th, 2017, 9.00-13.00

This exam consists of 11 numbered pages with 14 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: ____________________________________

STUDENT NUMBER: ____________________________________

There is only one correct answer to each question.

Problem 1 (10 %)

1. Is the compound proposition $\neg(p \vee \neg q) \leftrightarrow \neg p \wedge q$ a tautology?

   ☐ Yes ☐ No

2. Are the propositions $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ equivalent?

   ☐ Yes ☐ No

3. Is the compound proposition $(p \vee q) \rightarrow (p \wedge q)$ a tautology?

   ☐ Yes ☐ No

4. Is the compound proposition $(p \rightarrow q) \wedge (\neg p \rightarrow r)$ a tautology?

   ☐ Yes ☐ No

Problem 2 (4 %)

Which one of the following propositions is equivalent to

$$\neg(\forall x \exists y\ (x = y^2))$$

☐ $\exists x \forall y\ (x = y^2)$
☐ $\forall y \exists x\ (x = y^2)$
☐ $\forall x \exists y\ (x \neq y^2)$
☐ $\exists x \forall y\ (x \neq y^2)$
☐ $\exists y \forall x\ (x \neq y^2)$

Problem 3 (10 %)

Let A, B, C, D be sets, where

$$\begin{aligned} A &= \{x \in \mathbb{Z} \mid x \ \mathbf{mod}\ 9 = 0\}, \\ B &= \{x \in \mathbb{Z} \mid x \ \mathbf{mod}\ 3 = 0\}, \\ C &= \{x \in \mathbb{Z} \mid x \ \mathbf{mod}\ 15 = 0\}, \\ D &= \{x \in \mathbb{Z} \mid 0 \leq x \leq 17\}. \end{aligned}$$

1. Is $A \subseteq B$?

   ☐ Yes ☐ No

2. What is the cardinality of $B \cap D$?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7

3. What is the cardinality of the power set $\mathcal{P}(C \cap D)$?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 6 ☐ 8

4. What is the cardinality of $(A \cap D) \times (B \cap D)$?

   ☐ 0 ☐ 2 ☐ 4 ☐ 8 ☐ 10 ☐ 12 ☐ 14 ☐ 16

5. What is the cardinality of $(A \cap D) \cup (C \cap D)$?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 8

Problem 4 (9 %)

Let $f(x) = x \log(x^3 + 1) + x^2 \log x + x + \cos x + 123456$, for $x > 0$.
Answer the following true/false problems.

1. $f(x)$ is $O(x^3)$	☐ True	☐ False
2. $f(x)$ is $O(x \log x)$	☐ True	☐ False
3. $f(x)$ is $O(x^2 \log x)$	☐ True	☐ False
4. $f(x)$ is $\Omega(x^3)$	☐ True	☐ False
5. $f(x)$ is $\Omega(x \log x)$	☐ True	☐ False
6. $f(x)$ is $\Omega(x^2 \log x)$	☐ True	☐ False
7. $f(x)$ is $\Theta(x^3)$	☐ True	☐ False
8. $f(x)$ is $\Theta(x \log x)$	☐ True	☐ False
9. $f(x)$ is $\Theta(x^2 \log x)$	☐ True	☐ False

Problem 5 (5 %)

What is the value of 703^{13} **mod** 7?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

Problem 6 (5 %)

Consider the following algorithm:

procedure $Alg(n$: positive integer)
$a := 1$
$b := 1$
for $i := 1$ **to** n
  for $j := 1$ **to** n
    for $k := 1$ **to** n
      $a := 2 \cdot a$
    $b := 3 \cdot b$
$c := a + b$
return c

The number of multiplications used by this algorithm is

☐ $O(n)$ ☐ $\Theta(n^3)$ ☐ $O(n^2)$ ☐ $\Theta(n^5)$ ☐ $O(n^2 \log n)$

Problem 7 (4 %)

1. What is the value of $P(7, 2)$?

   ☐ 7 ☐ 14 ☐ 21 ☐ 35 ☐ 42 ☐ 49 ☐ 2520

2. What is the value of $C(7, 2)$?

   ☐ 7 ☐ 14 ☐ 21 ☐ 35 ☐ 42 ☐ 49 ☐ 2520

Problem 8 (8 %)

Let $p = 3$ and $q = 17$ and consider the RSA public key cryptosystem (page 295 in Rosen's book).

1. Which of the following numbers is not a valid encryption exponent?

   ☐ 5 ☐ 8 ☐ 11 ☐ 15 ☐ 21 ☐ 27

2. We encrypt the integer $M = 8$ using the encryption exponent $e = 3$. What is the encrypted message C?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7

3. What is the decrytion key d for the encryption exponent $e = 3$?

   ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14 ☐ 15 ☐ 16

Problem 9 (6 %)

Consider the following linear homogeneous recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}.$$

1. What is the degree of this recurrence relation?

   ☐ -1 ☐ 0 ☐ 1 ☐ 2 ☐ 3

2. Which of the following is the solution of this recurrence relation (α_1 and α_2 are constants)?

   ☐ $a_n = \alpha_1(-2)^n + \alpha_2$
   ☐ $a_n = \alpha_1 + \alpha_2 3^n$
   ☐ $a_n = \alpha_1 + \alpha_2 2^n$
   ☐ $a_n = \alpha_1 2^n + \alpha_2(-1)^n$
   ☐ $a_n = \alpha_1 2^n + \alpha_2 3^n$

Problem 10 (8 %)

Let $P(n)$ be the following statement

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

We want to prove by induction that $P(n)$ is true for all $n \geq 1$.

1. What is the correct basis step of the induction proof?

   ☐ Prove that $P(0)$ is true
   ☐ Prove that $P(1)$ is true
   ☐ Prove that $P(2)$ is true
   ☐ Prove that $P(n)$ is true, for all $n \geq 1$

2. Which one of the following is a correct outline of the inductive step?

   ☐ Let $k \geq 2$ and assume that $P(k)$ is true. By the induction hypothesis $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. Use this to prove $P(k+1)$.

   ☐ Let $k \geq 1$ and assume that $P(k)$ is true. By the induction hypothesis $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. Use this to prove $P(k+1)$.

   ☐ Let $k \geq 1$ and assume that $P(k)$ is true. By the induction hypothesis $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. Use this to prove $P(k)$.

   ☐ Let $k \geq 1$ and assume that $P(k)$ is true. By the induction hypothesis $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. Use this to prove $P(k+2)$.

   ☐ Let $k \geq -1$ and assume that $P(k)$ is true. By the induction hypothesis $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. Use this to prove $P(k+1)$.

Problem 11 (10 %)

Consider the relation R on the set $\{a, b, c, d\}$ given by the matrix

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

1. Answer the following true/false problems.

R is reflexive	☐ True	☐ False
R is symmetric	☐ True	☐ False
R is antisymmetric	☐ True	☐ False
R is transitive	☐ True	☐ False
R is an equivalence relation	☐ True	☐ False

2. What is the matrix that represents the relation R^2.

☐ $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

3. What is the matrix that represents the symmetric closure of R?

☐ $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

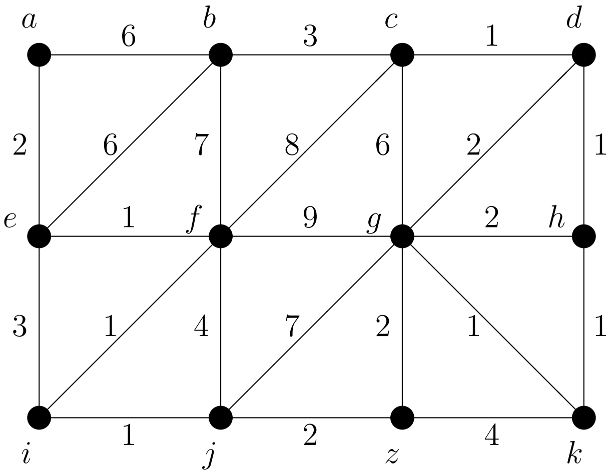

Figure 1: The graph G, considered in Problems 12, 13 and 14.

Problem 12 (10 %)

In this problem we use Dijkstra's algorithm (see Figure 2 on page 11) on the graph G in Figure 1.

1. What is the length of the shortest path from a to z (found by Dijkstra's algorithm)?

   ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14

2. In what order are vertices added to the set S ?

   ☐ a, e, f, j, z
   ☐ a, e, b, f, i, j, z
   ☐ $a, b, c, d, e, f, g, h, i, j, k, z$
   ☐ a, e, i, j, z
   ☐ a, e, f, g, z
   ☐ a, e, f, i, j, b, z
   ☐ a, e, f, i, j, z

Problem 13 (5 %)

What is the weight of a minimum spanning tree of the graph G in Figure 1.

☐ 14 ☐ 15 ☐ 16 ☐ 17 ☐ 18 ☐ 19 ☐ 20 ☐ 21

Problem 14 (6 %)

In this problem G is the graph in Figure 1. (The edge weights of G are not considered in this problem.)

1. Answer the following true/false problems.

G has an Euler circuit	☐ True	☐ False
G has an Euler path	☐ True	☐ False
G has a Hamilton circuit	☐ True	☐ False

2. What is the degree of the vertex b ?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7

3. What is the length of a shortest simple circuit of G?

   ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8

procedure *Dijkstra*(G: weighted connected simple graph, with
all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
$L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
other labels are ∞, and S is the empty set}
while $z \notin S$
$u :=$ a vertex not in S with $L(u)$ minimal
$S := S \cup \{u\}$
for all vertices v not in S
if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
{this adds a vertex to S with minimal label and updates the
labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

Figure 2: