Test Exam in Calculus

First Year at The Faculty of Engineering and Science
and The Faculty of Medicine

March 2016

The present exam set consists of 9 numbered pages with 14 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use books, notes etc. It **is not** allowed to use electronic devices.

Your answers must be marked directly in this exam set. The grading of the exam is based solely on these marks. Remember to write your **full name** and **student number** below.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Problem 1 (8 points)

A plane curve is given by

$$x = 1 + 2t^2,$$
$$y = 1 + t^3,$$

where the parameter t runs through the positive real numbers. Mark the correct statements below.

(a) (1 point). Which point on the curve corresponds to the parameter value $t = 1$?

☐ $(1,1)$ ☐ $(3,2)$ ☐ $(9,10)$ ☐ $(0,0)$ ☐ $(\frac{1}{4}, \frac{\sqrt{3}}{3})$

(b) (7 points). What is the curvature of the curve for $t = 1$?

☐ $\frac{12}{125}$ ☐ $\frac{24}{5}$ ☐ $\frac{\sqrt{3}}{2}$ ☐ 2 ☐ $\frac{12}{5}$

Problem 2 (6 points)

A curve in space is given by

$$x = \cos(2t),$$
$$y = \sin(2t),$$
$$z = t^2 + 3,$$

where the parameter t runs through the real numbers. Mark the correct expression for the arc length of the curve from $t = 0$ to $t = 3$.

☐ $\int_0^3 \sqrt{1 + 4t^2} dt$

☐ $\int_0^3 \sqrt{1 + (t^2 + 3)^2} dt$

☐ $\int_0^3 2\sqrt{1 + t^2} dt$

☐ $\int_0^3 (4 + t^2) dt$

☐ $\int_0^3 (2 + 2t) dt$

☐ $\int_0^3 (-\sin(2t) + \cos(2t) + 2t) dt$

Problem 3 (6 points)

A function is defined by

$$f(x) = \ln(1 + x^2).$$

Which of the polynomials below is the 2nd order Taylor polynomial for $f(x)$ about the point $a = 0$?

- ☐ $1 + x + \frac{1}{2}x^2$
- ☐ $x + \frac{1}{6}x^2$
- ☐ $x + x^2$
- ☐ x^2
- ☐ $x - \frac{1}{2}x^2$
- ☐ $\frac{1}{2}x^2 + \frac{1}{6}x^3$

Problem 4 (7 points)

Consider the differential equation

$$\frac{dy}{dx} = 6x^2y, \quad y > 0.$$

There is a unique solution $y(x)$ with initial value $y(0) = 3$. Answer the following questions regarding this solution:

(a) (3 points). What is the function value $y(1)$?

☐ $6e^3$ ☐ 12 ☐ $3e^2$ ☐ 6 ☐ $18e$

(b) (3 points). What is the value of $y'(1)$?

☐ $18e^2$ ☐ $36e^3$ ☐ 36 ☐ 2 ☐ $12\ln(6)$

(c) (1 points). What is the value of $y'(0)$?

☐ 0 ☐ 6 ☐ 12 ☐ 24 ☐ 3

Problem 5 (8 points)

Consider the initial value problem

$$\frac{dy}{dx} + 2xy = x, \quad y(0) = 1.$$

Answer the following questions regarding the unique solution $y(x)$:

(a) (4 points). What is the function value $y(1)$?

☐ $\frac{1}{2} + \frac{1}{2}e^{-1}$ ☐ e^{-1} ☐ 3 ☐ $1 + e^{-1}$ ☐ $1 + e$

(b) (4 points). What is the value of $y'(1)$?

☐ $-1 - 2e$ ☐ e^{-1} ☐ $-1 - 2e^{-1}$ ☐ $-e^{-1}$ ☐ -5

Problem 6 (7 points)

Consider the differential equation

$$y'' + 6y' + 10y = 0.$$

A number of function expressions, which contains two arbitrary constants c_1 and c_2, are listed below. Mark the expression which constitute the complete solution of the differential equation.

☐ $y(t) = c_1 e^{t} + c_2 e^{3t}$

☐ $y(t) = c_1 e^{2t} \cos(3t) + c_2 e^{2t} \sin(3t)$

☐ $y(t) = c_1 e^{3t} + c_2 t e^{3t}$

☐ $y(t) = c_1 e^{-2t} + c_2 e^{-4t}$

☐ $y(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)$

☐ $y(t) = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t)$

☐ $y(t) = c_1 e^{-t} + c_2 e^{-3t}$

☐ $y(t) = c_1 e^{-5t} + c_2 t e^{-5t}$

☐ $y(t) = c_1 e^{t} \cos(2t) + c_2 e^{t} \sin(2t)$

☐ $y(t) = c_1 e^{-3t} \cos(t) + c_2 e^{-3t} \sin(t)$

☐ $y(t) = c_1 e^{6t} + c_2 e^{10t}$

Problem 7 (6 points)

Consider the inhomogeneous differential equation

$$y'' + 4y' + 6y = 3t - 4.$$

(a) (3 points). Which of the following functions $y_p(t)$ can, for a suitable choice of the constants A and B, become a particular solution of the differential equation?

- ☐ $y_p(t) = Ae^{4t} + Be^{6t}$
- ☐ $y_p(t) = At + B$
- ☐ $y_p(t) = A\sin(4t) + B\cos(6t)$
- ☐ $y_p(t) = Ae^{3t} + Be^{-4t}$

(b) (3 points). What must the values of the constants A and B be in order to obtain a particular solution?

- ☐ $A = 1, \quad B = 3$
- ☐ $A = \frac{1}{5}, \quad B = -4$
- ☐ $A = -1, \quad B = 5$
- ☐ $A = \frac{1}{2}, \quad B = -1$

Problem 8 (8 points)

A function is defined by

$$f(x, y) = e^{4y+2xy}.$$

(a) (3 points). Mark the correct expression for the partial derivative $f_x(x, y)$.

- ☐ e^{2y}
- ☐ e^{4y+2xy}
- ☐ $2e^{2y}$
- ☐ $(4y + 2xy)e^{4y+2xy-1}$
- ☐ $6e^{4y+2xy}$
- ☐ $2ye^{4y+2xy}$

(b) (3 point). Mark the correct expression for the partial derivative $f_y(x, y)$.

- ☐ $(4 + 2x)e^{4y+2xy}$
- ☐ $4e^{4y+2xy}$
- ☐ e^{4+2x}
- ☐ $(4y + 2xy)e^{4y+2xy-1}$
- ☐ e^{4y+2xy}
- ☐ $2e^{4y+2xy}$

(c) (2 points). Which of the points below is a critical point of the function f?

- ☐ $(0, 0)$
- ☐ $(3, 0)$
- ☐ $(1, -2)$
- ☐ $(-1, 3)$
- ☐ $(2, 1)$
- ☐ $(-2, 0)$

Problem 9 (8 points)

A function is defined by

$$f(x, y) = x\sin(y) + x^2.$$

(a) (3 points). What is the gradient vector $\nabla f(P)$ at the point $P = (1, 0)$?

☐ $\mathbf{i} - \mathbf{j}$ ☐ $2\mathbf{i}$ ☐ $2\mathbf{i} + \mathbf{j}$ ☐ $-\mathbf{j}$ ☐ $\mathbf{0}$

(b) (2 points). What is the value of the directional derivative $D_{\mathbf{u}} f(P)$ at the point $P = (1, 0)$ and in the direction of the unit vector $\mathbf{u} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$?

☐ $\frac{8}{5}$ ☐ $\frac{1}{5}$ ☐ 7 ☐ $\frac{11}{5}$ ☐ $-\frac{3}{5}$

(c) (3 points). Which of the equations below describes the tangent plane to the graph of f at the point $Q = (1, 0, 1)$?

☐ $3x - y - z = 2$ ☐ $x - y + 2z = 3$

☐ $x - y - z = 0$ ☐ $x + y - z = \pi$

☐ $z = 1$ ☐ $2x + y - z = 1$

Problem 10 (10 points)

A region $\mathcal{R}$ in the plane consists of those points (x, y) which satisfy the inequalities

$$1 \le x^2 + y^2 \le 4, \quad 0 \le x, \quad 0 \le y.$$

Mark the value of the double integral

$$\iint_{\mathcal{R}} (x^2 + y^2 - 1)\, dA.$$

☐ $\frac{3\pi}{2}$ ☐ $\frac{11}{7}$ ☐ $\frac{9\pi}{8}$ ☐ $\frac{7\pi}{5}$ ☐ $\frac{15}{11}$

☐ 15 ☐ $\frac{\pi^2}{4}$ ☐ 10π ☐ 9 ☐ 21

Problem 11 (5 points)

Two complex numbers are given by

$$z_1 = \frac{2+i}{3-i} + \frac{3+i}{2}, \qquad z_2 = (e^{1+\frac{\pi}{3}i})^6.$$

(a) (3 points). What is z_1 written in standard form?

☐ $2+i$ ☐ $3-2i$ ☐ $-i$ ☐ $\frac{1}{5}-\frac{3}{5}i$ ☐ 0

(b) (2 points). What is z_2 written in standard form?

☐ 1 ☐ $-e^6$ ☐ e^3 ☐ $\frac{\sqrt{2}}{2}$ ☐ e^6

Remark. In problem 12 the evaluation of your answers will be performed using the following principle: Every false mark cancels a true mark.

Problem 12 (6 points)

Let $\mathcal{T}$ be the region in space consisting of those points (x, y, z) which satisfy the inequalities

$$-2 \le x \le 2, \quad 0 \le y \le 4-x^2, \quad -y \le z \le y.$$

A solid body with density function $\delta(x,y,z) = 3-y$ covers region $\mathcal{T}$ precisely. The volume of the body is denoted V and its mass is denoted m. Mark all of the correct expressions below. Note: You are **not** asked to evaluate the integrals.

☐ $m = \int_{-2}^{2} \int_{0}^{4-x^2} \int_{-y}^{y} (3-y)\, dz dy dx.$

☐ $m = \int_{0}^{4} \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-y}^{y} (3-y)\, dz dx dy.$

☐ $m = \int_{0}^{4-x^2} \int_{-2}^{2} \int_{-y}^{y} (3-y)\, dz dx dy.$

☐ $V = \int_{0}^{4-x^2} \int_{-2}^{2} \int_{-y}^{y} dz dx dy.$

☐ $V = \int_{-2}^{2} \int_{0}^{4-x^2} \int_{-y}^{y} dz dy dx.$

Problem 13 (10 points).

Answer the following 5 True/False problems:

(a) (2 points). One has

$$\arctan\big(\tan\big(\frac{5\pi}{4}\big)\big) = \frac{5\pi}{4}.$$

☐ True ☐ False

(b) (2 points). Let D be the region in the plane consisting of those points (x, y) which satisfy the inequalities $0 \leq x \leq 1$ and $0 < y \leq 1$. Let f be the function with rule

$$f(x, y) = x^2 y + \frac{1}{y}$$

and domain D. Then f attains a global maximum on D.

☐ True ☐ False

(c) (2 points). The domain of the function $f(x, y) = \sqrt{x^2 + y^2}$ equals $\mathbf{R}^2$ corresponding to the entire xy-plane.

☐ True ☐ False

(d) (2 points). For the function $f(x, y) = \sqrt{x^2 + y^2}$, the partial derivative $f_x(x, y)$ does not exist at the point $(x, y) = (0, 0)$.

☐ True ☐ False

(e) (2 points). The point with rectangular coordinates $(x, y) = (1, 1)$ can be described by $(r, \theta) = (-\sqrt{2}, \frac{5\pi}{4})$ in polar coordinates.

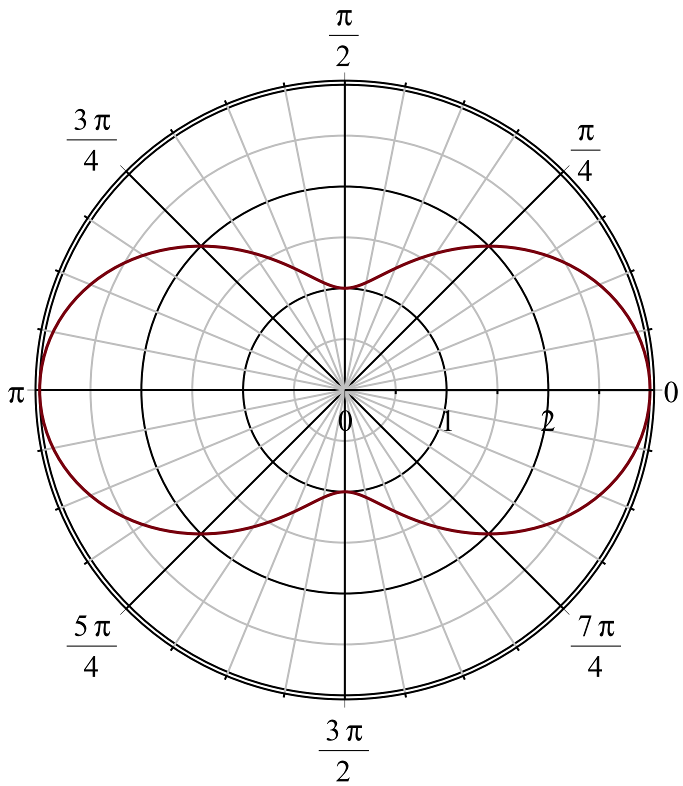
☐ True ☐ False

Problem 14 (5 points)

The figure below shows the graph of the function

$$r = f(\theta), \quad 0 \leq \theta \leq 2\pi$$

in polar coordinates.



Which one of the following rules for f corresponds to this figure?

- ☐ $f(\theta) = 2 - \cos(3\theta)$
- ☐ $f(\theta) = 2 + \cos(\theta)$
- ☐ $f(\theta) = 1 + \sin(3\theta)$
- ☐ $f(\theta) = 2 + \cos(2\theta)$
- ☐ $f(\theta) = 2 + \sin(2\theta)$
- ☐ $f(\theta) = 1 + \sin(\theta)$