Reexam in Calculus

Wednesday 17 February 2016

First Year at the Faculty of Engineering and Science
and
The Faculty of Medicine

The present exam set consists of 7 numbered pages with 12 problems.

It is allowed to use books, notes etc. It is **not allowed** to use **electronic devices**.

The listed percentages specify the weightings of the individual problems in the assessment of the exam.

The exam set has two independent parts.

- Part I contains "regular problems”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II consists of "multiple choice” problems. **The answers of Part II** must be given on **these sheets of paper**.

Remember to write your full name and student number on each page of your solutions. **Number each page** and indicate the **total number of sheets** on page 1. Good luck!

NAME: ____________________

STUDENT NUMBER: ____________________

Part I ("regular problems")

Problem 1 (8%)

A plane curve is given by

$$x = e^{t},$$
$$y = e^{-t},$$

where the parameter t runs through the real numbers.

(a) Find an expression for the curvature $\kappa(t)$ of the curve.

(b) Verify that the point $P = (1,1)$ lies on the curve and compute the curvature of the curve at this point.

Problem 2 (7%)

Find the Taylor polynomial of degree two for the function

$$f(x) = \sqrt{2x+1}$$

about the point $x = 0$.

Problem 3 (10%)

(a) Find the general solution of the differential equation

$$y'' + 3y' + 2y = 0.$$

(b) Find the general solution of the inhomogeneous differential equation

$$y'' + 3y' + 2y = 4t + 8.$$

Problem 4 (8%)

Consider the function

$$f(x,y) = \arctan\left(\frac{x+y}{4}\right).$$

(a) Find the domain of f.

(b) Sketch the level curve with the equation $f(x,y) = 1$.

Problem 5 (8%)

A function is defined by

$$f(x,y,z) = \sin(2x+y) + z^2.$$

(a) Find the gradient vector $\nabla f(x,y,z)$.

(b) In which direction from the point $P = (1,-2,1)$ does f increase the most? (Specify a unit vector).

Problem 6 (12%)

The surface $\mathscr{F}$ in space is determined by the equation $z = f(x,y)$ where

$$f(x,y) = \ln(x^2 + y).$$

(a) Verify that the point $P = (2,-3,0)$ lies on the surface $\mathscr{F}$.

(b) Determine the partial derivatives $f_x(x,y)$ and $f_y(x,y)$.

(c) Find an equation of the tangent plane to the surface $\mathscr{F}$ at the point $P = (2,-3,0)$.

Problem 7 (12%)

Let $\mathscr{R}$ be the plane region consisting of those points (x, y) for which

$$1 \leq x^2 + y^2 \leq 9.$$

(a) Sketch the region $\mathscr{R}$.

(b) Evaluate the double integral

$$\iint_{\mathscr{R}} \frac{1}{x^2 + y^2} \, dA.$$

Problem 8 (7%)

Find the complex roots of the polynomial

$$z^2 - (3 + 4i)z - 1 + 7i.$$

Part II ("multiple choice” problems)

Problem 9 (10%).

Answer the following 5 True/False problems:

a. For every real number b the following equation, in the unknown x, has precisely one solution:

$$\arcsin(x) = b.$$

☐ True ☐ False

b. For the function $f(t) = e^{(1+2i)t}$, where t is a real variable, one has

$$\frac{f'(t)}{f(t)} = 1 + 2i.$$

☐ True ☐ False

c. Let D be the plane region consisting of those points (x, y) for which $-1 \le x \le 1$ and $0 \le y \le 1$. Let f be the function with rule

$$f(x, y) = \sin(x + y) + x^3$$

and domain D. Then f attains a global minimum on D.

☐ True ☐ False

d. For every complex number z one has

$$z\bar{z} \ge 0.$$

☐ True ☐ False

e. The point having rectangular coordinates $(x, y) = (1, 2)$ can be represented by the polar coordinates $(r, \theta) = (\sqrt{5}, \frac{\pi}{4})$.

☐ True ☐ False

Remark. In problems 10 and 11 the evaluation of your answers will be performed using the following principle: Every false mark cancels a true mark.

Problem 10 (6%)

Let $\mathscr{T}$ be the region in space consisting of those points (x, y, z) for which

$$0 \le x \le 1, \quad 0 \le y \le 1 - x, \quad 0 \le z \le 1 - x^2 - y^2.$$

A solid body with density function $\delta(x, y, z) = 3 - x^2 - y^2 - z^2$ covers precisely this region. The volume of the body is denoted V and its mass is denoted m. Mark all correct expressions below. Remark: **Do not** evaluate the integrals.

☐ $m = \int_0^1 \int_0^{1-x} \int_0^{1-x^2-(1-x)^2} (3 - x^2 - y^2 - z^2)\, dz dy dx.$

☐ $m = \int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} (3 - x^2 - y^2 - z^2)\, dz dy dx.$

☐ $m = \int_0^1 \int_{1+y}^{1-y} \int_0^{1-x^2-y^2} (3 - x^2 - y^2 - z^2)\, dz dx dy.$

☐ $V = \int_0^1 \int_0^{1-y} \int_0^{1-x^2-y^2} dz dx dy.$

☐ $V = \int_0^1 \int_0^{2} \int_0^{1-x^2-y^2} dz dy dx.$

Problem 11 (6%)

A complex polynomial is given by

$$p(z) = z^3 + 5z^2 + z + 5.$$

Mark all correct statements below.

☐ The complex number i is a root of the polynomial $p(z)$.

☐ $p(z)$ has no real roots.

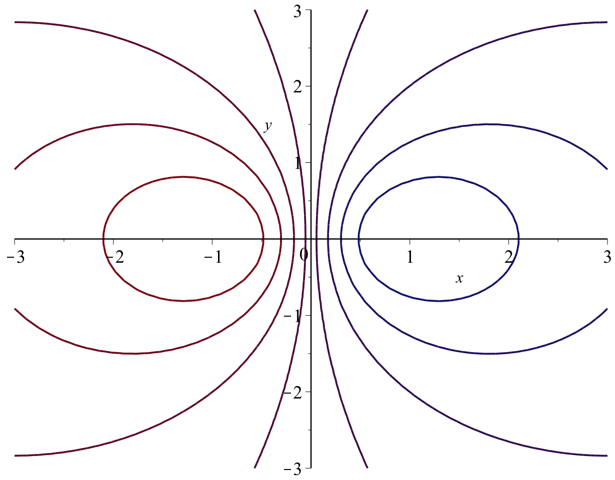
☐ $p(z)$ has precisely 3 roots, counting multiplicity.

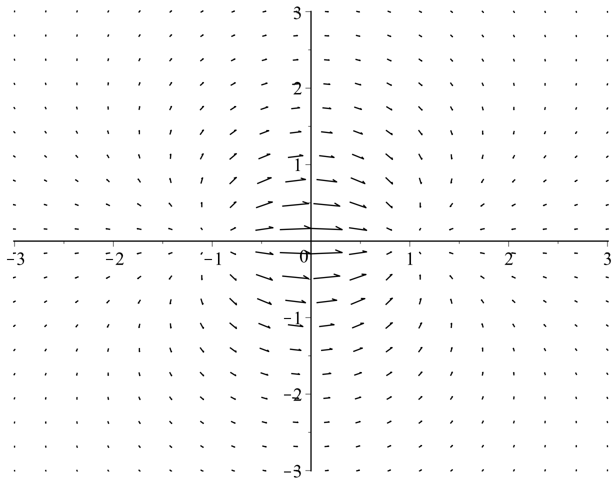
☐ $p(z)$ has three different roots.

☐ $p(z)$ has a root of multiplicity 3.

Problem 12 (6%)

A differentiable function $f(x, y)$ is defined for $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$. The two figures below show selected level curves and gradient vectors for the function. There are precisely two critical points with coordinates $(-1, 0)$ and $(1, 0)$. Determine the type of each critical point and mark your answer below.





a. At $(-1, 0)$ the function f has a

- ☐ local maximum
- ☐ local minimum
- ☐ saddle point

b. At $(1, 0)$ the function f has a

- ☐ local maximum
- ☐ local minimum
- ☐ saddel point