Reexam in Calculus

Tuesday 18 August 2015

First Year at the Faculty of Engineering and Science
and
The Faculty of Medicine

The present exam set consists of 7 numbered pages with 12 problems.

It is allowed to use books, notes etc. It is **not allowed** to use **electronic devices**.

The listed percentages specify the weightings of the individual problems in the assessment of the exam.

The exam set has two independent parts.

- Part I contains "regular problems”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II consists of "multiple choice” problems. **The answers of Part II** must be given on **these sheets of paper**.

Remember to write your full name and student number on each page of your solutions. **Number each page** and write the **total number of pages** on the front page. Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Part I ("regular problems”)

Problem 1 (8%)

A plane curve is given by

$$x = t^3 + 1,$$
$$y = 4t - 1,$$

where $t \in \mathbb{R}$.

(a) Find the curvature $\kappa(t)$ of the curve for $t \in \mathbb{R}$.

(b) Verify that $P = (2,3)$ is a point on the curve and compute the curvature of the curve at this point.

Problem 2 (10%)

(a) Find the general solution of the differential equation

$$y'' + y' - 2y = 0.$$

(b) Solve the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 3, \quad y'(0) = 0.$$

Problem 3 (7%)

Find the Taylor polynomial of degree 4 for the function

$$f(x) = e^x + e^{-x}$$

about the point $a = 0$.

Problem 4 (8%)

Consider the function

$$f(x,y) = \frac{x+y}{x-y}.$$

(a) Find the domain of f.

(b) Sketch the level curve with the equation $f(x,y) = -2$.

Problem 5 (12%)

A function is defined by

$$f(x,y) = \ln(x^2 + y^2 + 2).$$

(a) Find the gradient vector $\nabla f(x,y)$.

(b) Compute the directional derivative of f at the point $P = (1,-1)$ in the direction of the unit vector

$$\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

(c) Find a critical point for the function f.

Problem 6 (12%)

A plane region is given by

$$\mathscr{R} = \left\{(x,y) : \quad 1 \le x^2 + y^2 \le 4\right\}.$$

(a) Sketch the region $\mathscr{R}$.

(b) Evaluate the double integral

$$\iint_{\mathscr{R}} \sqrt{x^2+y^2}\, dA.$$

Problem 7 (7%)

Find all complex roots of the polynomial

$$z^2 - 3z + 3 - i.$$

Problem 8 (8%)

The surface $\mathscr{F}$ in space is determined by the equation

$$z = \arctan(x - y).$$

(a) Verify that the point $P = (2, 1, \frac{\pi}{4})$ lies on the surface $\mathscr{F}$.

(b) Find an equation of the tangent plane to the surface $\mathscr{F}$ at the point $P = (2, 1, \frac{\pi}{4})$.

Part II ("multiple choice” problems)

Problem 9 (10%).

Answer the following 5 True/False problems:

a. For every complex number z one has

$$z + \overline{z} = 2\text{Re}(z).$$

☐ True ☐ False

b. The following identity holds for all $x \in \mathbb{R}$:

$$\sin(2x) = 2\sin(x).$$

☐ True ☐ False

c. Let $f(x, y)$ be a function defined on $\mathbb{R}^2$. If $f_x(a, b) = 0$ and $f_y(a, b) = 0$ then $f(x, y)$ attains a local maximum or local minimum at the point (a, b).

☐ True ☐ False

d. If x is a real number and $\tan(x) \neq 0$ then

$$\tan(x)\arctan(x) = 1.$$

☐ True ☐ False

e. One has the following identity:

$$e^{2\pi i} = 1.$$

☐ True ☐ False

Remark. In problems 10 and 11 the evaluation of your answers will be performed using the following principle: Every false mark cancels a true mark.

Problem 10 (7%)

A solid body $\mathscr{T}$ covers precisely the region in space given by

$$\{(x,y,z): \quad (x,y) \in \mathscr{R}, \quad y \le z \le 2y\}$$

where

$$\mathscr{R} = \{(x,y): \quad -1 \le x \le 1, \quad x^2+1 \le y \le 2\}.$$

The density function of $\mathscr{T}$ is $\delta(x,y,z) = z$. Mark all correct statements below. Remark: **Do not** evaluate the integrals.

- ☐ The mass of $\mathscr{T}$ can be calculated as follows: $m = \int_{-1}^{1}\int_{x^2+1}^{2}\int_{y}^{2y} z\,dz\,dx\,dy.$
- ☐ The mass of $\mathscr{T}$ can be calculated as follows: $m = \int_{-1}^{1}\int_{x^2+1}^{2}\int_{y}^{2y} z\,dz\,dy\,dx.$
- ☐ The volume of $\mathscr{T}$ can be calculated as follows: $V = \int_{1}^{2}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{y}^{2y} dz\,dx\,dy.$
- ☐ The volume of $\mathscr{T}$ can be calculated as follows: $V = \int_{1}^{2}\int_{-\sqrt{y-1}}^{\sqrt{y-1}}\int_{y}^{2y} dz\,dx\,dy.$
- ☐ The volume of $\mathscr{T}$ can be calculated as follows: $V = \int_{-1}^{1}\int_{x^2+1}^{2}\int_{y}^{2y} dz\,dy\,dx.$

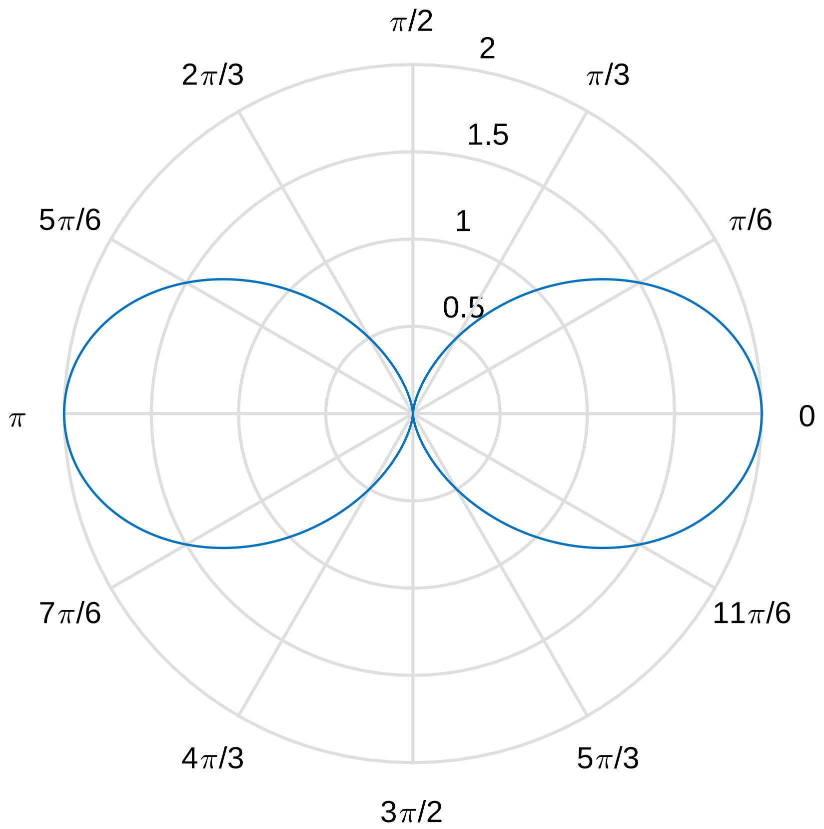
Problem 11 (6%)

Let $p(z)$ be a complex polynomial of degree 7. Suppose that $p(z)$ has real coefficients. Mark all correct statements below.

- ☐ $p(z)$ will always have 7 complex roots, counting multiplicity.
- ☐ There will always exist a degree 1 polynomial $q_1(z)$ with real coefficients and a degree 6 polynomial $q_2(z)$ such that $p(z) = q_1(z)q_2(z)$.
- ☐ If $z_0 \ne 0$ is a complex root of $p(z)$ then z_0^{-1} is also a complex root of $p(z)$.
- ☐ $p(z)$ will always have 7 real roots, counting multiplicity.
- ☐ $p(z)$ will always have at least one real root.

Problem 12 (5%)

The figure below shows the graph of the function $r = f(\theta)$ displayed in polar coordinates.



Which one of the below expressions of f along with domain indication for θ corresponds to the figure above?

- ☐ $f(\theta) = 2 - \cos(3\theta),\ 0 \le \theta \le \frac{\pi}{3}$
- ☐ $f(\theta) = 1 + \sin(\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 + \cos(4\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 + \cos(2\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 + \sin(2\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 - \sin(2\theta),\ 0 \le \theta \le \pi$