Re-Exam in Calculus

Tuesday, august 20, 2013

First Year at The TEK-NAT Faculty and Health Faculty

The present exam consists of 7 numbered pages with a total of 12 exercises.

It is allowed to use books, notes, etc. It is **not** allowed to use electronic devices.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts.

- Part I contains “regular exercises”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II is “multiple choice” exercises. The answers for Part II must be given on these sheets

Remember to write your full name (including middle names) and student number on each page of your answers. **Number each page and write the total number of pages on the front page of the answers.**

Good luck!

NAME: ________________________________

STUDENT NUMBER: ________________________________

Part I ("regular exercises")

Excercise 1 (12%)

(a) Find the complete solution to the differential equation

$$y'' + 2y' - 5y = 0.$$

(b) Find the complete solution to the inhomogeneous differential equation

$$y'' + 2y' - 5y = 3t + 1.$$

Excercise 2 (7%)

Determine the Taylor polynomial of degree 3 for

$$f(x) = \sin(2x) - x,$$

around $a = 0$.

Excercise 3 (8%)

A surface $\mathscr{F}$ is given by $z = f(x, y)$, with

$$f(x, y) = x^2 + y^4.$$

Determine an equation for the tangent plane to $\mathscr{F}$ through the point $P(2, 1, f(2, 1))$.

Excercise 4 (8%)

Find all complex solution to the equation

$$z^2 - (1+3i)z - 2 + 2i = 0.$$

Excercise 5 (16%)

(a) The region $\mathscr{R}$ in the xy-plane is bounded by the curves $y = x$ and $y = \sqrt{x}$. Sketch $\mathscr{R}$.

(b) Compute the integral of $f(x, y) = x^2$ over the planar region $\mathscr{R}$.

(c) A body T with density $\delta(x, y, z) = x^2$ covers exactly the region in space given by

$$\{(x, y, z) : (x, y) \in \mathscr{R}, 0 \le z \le x\}.$$

Determine the mass of T.

Excercise 6 (8%)

Consider the function

$$f(x, y, z) = 3z^2 + \sqrt{x^2 + y^4},$$

defined on $\{(x, y, z) : x > 0, y > 0, z > 0\}$.

(a) Calculate the gradient $\nabla f(x, y, z)$.

(b) Find the direction of fastest increase of f at the point $P(2, 1, 0)$?

Excercise 7 (5%)

Consider the function

$$f(x,y) = \sin(x^2) + \cos(xy).$$

(a) Determine the domain of f.

(b) Find the partial derivatives $f_x(x,y)$ og $f_y(x,y)$.

Excercise 8 (8%)

Consider the parametric curve given by

$$\mathbf{r}(t) = \frac{t^3}{3}\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + t\,\mathbf{k} = \begin{bmatrix} t^3/3 \\ t^2/\sqrt{2} \\ t \end{bmatrix}, \qquad t \in \mathbb{R}.$$

Compute the arc length of the curve from $t = 0$ til $t = 3$. Hint: $(1+a)^2 = 1 + a^2 + 2a$.

Del II ("multiple choice" opgaver)

Notice. In exercises 9 and 10 the evaluation is performed using the following principle: Every false mark cancels one true mark. Therefore, one gains *nothing* from a "safe bet".

Excercise 9 (6%)

A body T covers the area in space that is given by

$$\{(x,y,z): \quad (x,y) \in R, y \le z \le \sqrt{x} + 2y^2\},$$

with

$$R = \{(x,y): \quad 0 \le x \le 1, x^2 \le y \le \sqrt{x}\}.$$

The density of T is $\delta(x,y,z) = xz^2$. Mark all correct statements below (note: The value of the integral should *not* be computed).

- ☐ The mass of T can be computed as follows: $m = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_y^{\sqrt{x}+2y^2} xz^2 \, dz dy dx$
- ☐ The volume of T can be computed as follows: $V = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_y^{\sqrt{x}+2y^2} z^2 \, dz dy dx$
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- ☐ The mass of T can be computed as follows: $m = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_y^{\sqrt{x}+2y^2} x^2 z^4 \, dz dy dx$
- ☐ The volume of T can be computed as follows: $V = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_y^{\sqrt{x}+2y^2} dz dy dx.$

Excercise 10 (5%)

Consider a complex polynomial $p(z)$ of degree 5. Mark all correct statements below.

- ☐ There exists $r_0 \in \mathbb{R}$ such that $p(z) = (z - r_0)q(z)$ with $q(z)$ a complex polynomial polynomial of degree 4.
- ☐ $p(z)$ has exactly 5 different complex roots.forskellige komplekse rødder
- ☐ $p(z)$ has exactly 5 complex roots, counting multiplicity
- ☐ There exists $z_0 \in \mathbb{C}$, such that $p(z) = (z - z_0)q(z)$ with $q(z)$ a complex polynomial of degree 4.

Excercise 11 (10%).

Answer the following 6 true/false exercises:

a. The arc length function $s(t)$ for a differentiable curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ is defined as

$$s(t) = \int_0^t |\mathbf{r}'(\tau)|^2 \, d\tau, \qquad t \in \mathbb{R}.$$

☐ True ☐ False

b. A complex polynomial

$$P(z) = a_n z^2 + \cdots + a_1 z + a_0,$$

with $a_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, can be factored as a product of real polynomial of degree 1 and real polynomial of degree 2.

☐ True ☐ False

c. Der gælder, at

$$\frac{d}{dx} \sin^{-1}(2x) = \frac{2}{\sqrt{1-4x^2}}, \qquad x \in (-1/2, 1/2).$$

☐ True ☐ False

d. A surjective (onto) function $f : \mathbb{R} \to \mathbb{R}$ always has an inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$.

☐ True ☐ False

e. It holds true that

$$\cos\big(\arccos(x)\big) = x$$

for *all* $x \in \mathbb{R}$.

☐ True ☐ False

f. It holds true that

$$\sin(4x) = 2\sin(2x)\cos(2x)$$

for *all* $x \in \mathbb{R}$.

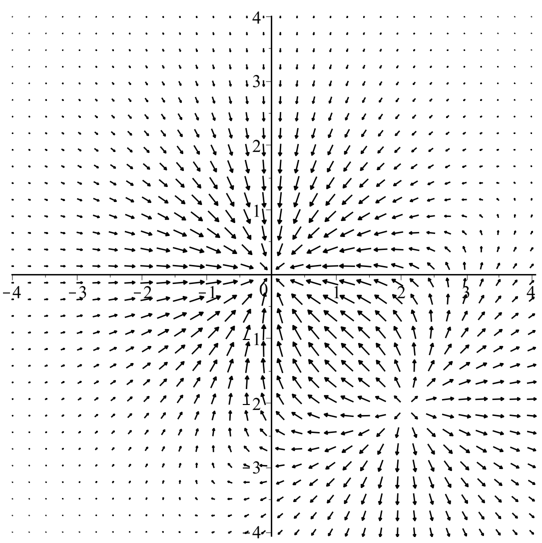
☐ True ☐ False

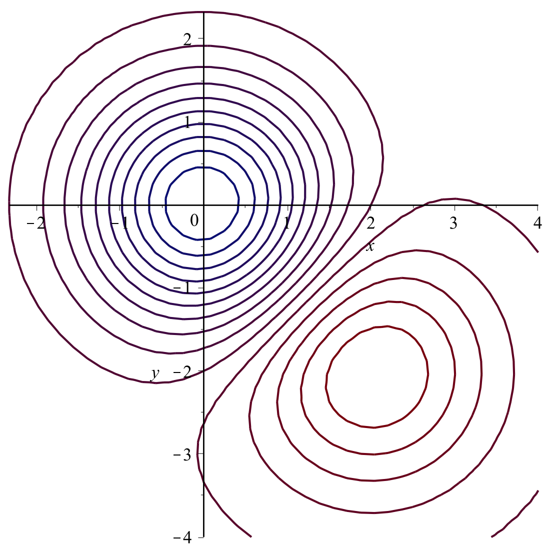
Excercise 12 (7%)

A function $f(x,y)$ is defined on the square

$$R = \{(x,y) : -4 \leq x, y \leq 4\}.$$

The two figures below show gradient vectors and level curves, respectively, for the function in R. The function has two critical points in R with coordinates $(0,0)$ and $(2,-2)$. Determine the type of critical points from the plot, and mark your answers below.





(a) The point $(0,0)$ is

- ☐ a local max
- ☐ a locan min
- ☐ a saddle point.

(b) The point $(2,-2)$ is

- ☐ a local max
- ☐ a locan min
- ☐ a saddle point.