Re-exam in Calculus

Thursday August 11th 2011

First Year at The TEK-NAT Faculty and Health Faculty

The present exam consists of 7 numbered pages with a total of 12 exercises.

It is allowed to use books, notes, etc. It is **not** allowed to use electronic devices.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts.

- Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II is "multiple choice" exercises. The answers for Part II must be given on these sheets

Remember to write your full name (including middle names), student number together with the course number on each side of your answers. **Number each page and write the total number of pages on the front page of the answers.**

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

COURSE NUMBER: ______________________________

Part I ("regular exercises")

Exercise 1 (8%)

The area T in space are bounded by the surfaces $z = (x^2 + y^2)^2$ and $z = 1$.

- Determine the volume of T [Hint: Cylinder coordinates].

Exercise 2 (8%)

The function $f(x)$ has the property that its Taylor polynomial of degree 3 around $a = a$ is

$$P_3(x) = 2 - x + 3x^2 - x^3.$$

(a) Determine the value $f(0)$.

(b) Determine the value $f'(0)$.

(c) Determine the value $f''(0)$.

Exercise 3 (8%)

Consider

$$f(x,y) = x^3y - 2xy^2.$$

(a) Determine the partial derivative $\frac{\partial f}{\partial x}(x,y)$

(b) Determine the partial derivative $\frac{\partial f}{\partial y}(x,y)$

(c) Determine an equation for the tangent plane to the surface $z = f(x,y)$ through the point $(1,1,-1)$.

Exercise 4 (10%)

Determine the double integral of $f(x,y) = x^2 + y^2$ over the area

$$R = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}.$$

Exercise 5 (7%)

Solve the equation

$$z^3 = 8,$$

where the solutions must be specified on the form $a + ib$, with $a, b \in \mathbb{R}$.

Exercise 6 (8%)

There is given a complex number $z = 1 + i$.

(a) Determine the modulus of z^4.

(b) Determine the modulus of z^{-4}.

(c) Write the complex number z^4 in polar form.

Exercise 7 (8%)

Consider the function

$$f(x, y, z) = xy^2z^3.$$

(a) Determine the gradient vector $\nabla f(x, y, z)$.

(b) Determine the directional derivative of f in the point $P(1, 1, 1)$ in the direction given by $\mathbf{v} = \mathbf{i} + 4\mathbf{k}$.

Exercise 8 (12%)

(a) Find the complete solution to the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$$

(b) Find the complete solution to the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = x + 1.$$

Part II (”multiple choice” exercises)

Notice. In exercises 9 and 10 the evaluation is performed using the following principle: Every false mark cancels one true mark. Therefore, one gains *nothing* from a “safe bet”.

Exercise 9 (7%)

Consider a complex polynomial $p(z)$ of degree 5 with *real coefficients*. Mark all correct statements below.

- ☐ There exists a $a \in \mathbb{C}$, such that $p(z) = (z - a)q(z)$ where $q(z)$ is a complex polynomial of degree 4
- ☐ $q(z)$ can *always* be factorized as a product consisting only of 1. and 2. degree polynomials with real coefficients
- ☐ $q(z)$ always contain at least one linear real factor
- ☐ $q(z)$ always has 5 different complex roots
- ☐ $q(z)$ can be factorized using only 1. degree polynomials with real coefficients
- ☐ $p(z)$ might not have any roots in the complex plane

Exercise 10 (7%)

Consider a function $f(x,y)$ of two variables defined on $\mathbb{R}^2$. We are informed that the partial derivatives f_x and f_y exists and are continuous in an neighborhood of the point $P(a,b)$. Mark all correct statements below.

☐ f is continuous in $P(a,b)$.

☐ All directional derivatives $D_{\mathbf{u}}f(a,b)$ exist.

☐ It is always true that

$$\lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a}=0.$$

☐ f is differentiable in $P(a,b)$.

☐ It is not necessarily true that $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$.

Exercise 11 (10%).

Answer the following 5 true/false exercises:

a. For two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, with $a_1, a_2, b_1, b_2 \in \mathbb{R}$, it is always true that

$$z_1 z_2 = a_1 a_2 + b_1 b_2 + i(a_1 b_2 + b_1 a_2).$$

☐ True ☐ False

b. We have that

$$\arccos(\cos x)) = x, \qquad \text{for alle } x \in [0, \pi].$$

☐ True ☐ False

c. Let a, b and c be real constants. Then the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = x^3,$$

has exactly one solution.

☐ True ☐ False

d. The function $f(x, y, z)$ is defined on $\mathbb{R}^3$, and all directional derivatives f exist in the point $P(0, 0, -3)$. Then all partial derivatives of f of order 1 exists in $P(0, 0, -3)$.

☐ True ☐ False

e. A complex polynomial of degree n, $n \geq 1$, has exactly n different complex roots.

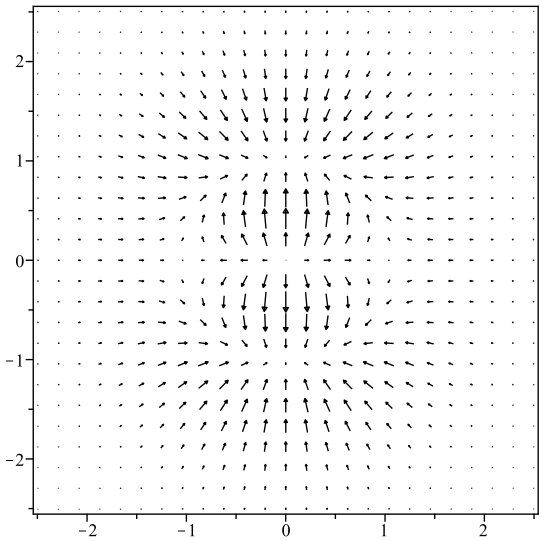
☐ True ☐ False

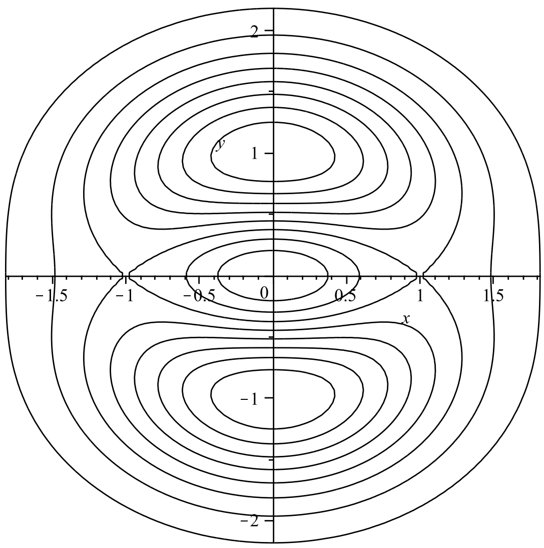
Exercise 12 (6%)

A function $f(x,y)$ is defined on the square

$$R = \{(x,y) : -5/2 \leq x, y \leq 5/2\}.$$

The two figures below show certain gradient vectors and level curves, respectively, of the function on R. The function has three critical points in R with coordinates $(0,0)$ and $(0,\pm 1)$. Use the figures to determine the nature of each of the critical points and mark the answer below.





(a) In the point $(0,0)$ f has a:

- ☐ local maximum
- ☐ local minimum
- ☐ saddle point.

(b) In the point $(0,-1)$ f has a:

- ☐ local maximum
- ☐ local minimum
- ☐ saddle point.

(c) In the point $(0,1)$ f has a:

- ☐ local maximum
- ☐ local minimum
- ☐ saddle point.