Exam in Calculus

Friday 8 January 2016

First Year at the Faculty of Engineering and Science
and
The Faculty of Medicine

The present exam set consists of 7 numbered pages with 12 problems.

It is allowed to use books, notes etc. It is **not allowed** to use **electronic devices**.

The listed percentages specify the weightings of the individual problems in the assessment of the exam.

The exam set has two independent parts.

- Part I contains "regular problems”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II consists of "multiple choice” problems. **The answers of Part II** must be given on **these sheets of paper**.

Remember to write your full name and student number on each page of your solutions. **Number each page** and indicate the **total number of sheets** on page 1. Good luck!

NAME: ________________________________

STUDENT NUMBER: ________________________________

Part I ("regular problems")

Problem 1 (8%)

A particle is moving along a curve in space. The coordinates of the particle at time t is given by

$$x = 4t + 1,$$
$$y = \cos(3t),$$
$$z = \sin(3t).$$

(a) Determine the arc length of the curve from $t = 0$ to $t = 4$.

(b) Find the acceleration vector for the particle.

Problem 2 (6%)

Let $f(x)$ be a function where the first 3 derivatives exist at $x = 0$. The Taylor polynomial of degree 3 for $f(x)$ at $a = 0$ is

$$P_3(x) = 2 - 7x + x^3.$$

Find the values $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$.

Problem 3 (10%)

(a) Find the general solution of the differential equation

$$y'' + 4y' + 5y = 0.$$

(b) Solve the initial value problem

$$y'' + 4y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 3.$$

Problem 4 (8%)

Consider the function

$$f(x,y) = \frac{5x+3y-1}{x+y}.$$

(a) Find the domain of f.

(b) Sketch the level curve with the equation $f(x,y) = 2$.

Problem 5 (12%)

A function is defined by

$$f(x,y) = xe^{y}.$$

(a) Find the gradient vector $\nabla f(x,y)$.

(b) Compute the directional derivative of f at the point $P = (1,0)$ in the direction of the unit vector

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

(c) In which direction is the directional derivative at $P = (1,0)$ maximal? (Specify a unit vector). In which direction is the directional derivative at P minimal?

Problem 6 (9%)

The surface $\mathscr{F}$ in space is determined by the equation $F(x,y,z) = 0$ with

$$F(x,y,z) = x^2 + \sin(xy) + 4z.$$

(a) Verify that the point $P = (2,0,-1)$ lies on the surface $\mathscr{F}$.

(b) Determine the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$.

(c) Find an equation of the tangent plane to the surface $\mathscr{F}$ at the point $P = (2,0,-1)$.

Problem 7 (9%)

Let $\mathscr{R}$ be the plane region consisting of those points (x, y) for which

$$0 \le x, \quad x^2 + y^2 \le 4.$$

(a) Sketch the region $\mathscr{R}$.

(b) Evaluate the double integral

$$\iint_{\mathscr{R}} e^{x^2+y^2}\, dA.$$

Problem 8 (7%)

Find the complex roots of the polynomial

$$z^2 + (-3 + i)z + 4 - 3i.$$

Part II ("multiple choice" problems)

Problem 9 (10%).

Answer the following 5 True/False problems:

a. The point having rectangular coordinates $(x, y) = (1, 1)$ can be represented by the polar coordinates $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$.

☐ True ☐ False

b. The following identity holds:

$$e^{3\pi i} = -3.$$

☐ True ☐ False

c. The function

$$f(x, y) = 2x^2 - 3y^2$$

attains a global maximum at the point $(0, 0)$.

☐ True ☐ False

d. The following identity holds:

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{\pi}{4}.$$

☐ True ☐ False

e. For every real number θ one has

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta.$$

☐ True ☐ False

Remark. In problems 10 and 11 the evaluation of your answers will be performed using the following principle: Every false mark cancels a true mark.

Problem 10 (6%)

Let $\mathscr{T}$ be the region in space consisting of those points (x, y, z) for which

$$-1 \le x \le 1, \quad 0 \le y \le \sqrt{1-x^2}, \quad -y \le z \le y.$$

A solid body with density function $\delta(x, y, z) = 2 - y^2$ covers precisely this region. The volume of the body is denoted V and its mass is denoted m. Mark all correct formulas below. Remark: **Do not** evaluate the integrals.

☐ $V = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{-y}^{y} dz\,dy\,dx.$

☐ $V = \int_{0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-y}^{y} dz\,dx\,dy.$

☐ $V = \int_{-y}^{y} \int_{0}^{\sqrt{1-x^2}} \int_{-1}^{1} dx\,dy\,dz.$

☐ $m = \int_{-y}^{y} \int_{0}^{\sqrt{1-x^2}} \int_{-1}^{1} (2-y^2)\,dx\,dy\,dz.$

☐ $m = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{-y}^{y} (2-y^2)\,dz\,dy\,dx.$

Problem 11 (6%)

Let $p(z)$ be a complex polynomial of degree 9. Suppose that $p(z)$ has real coefficients. Mark all correct statements below.

☐ $p(z)$ will always have at least one real root.

☐ $p(z)$ will always have at least two real roots.

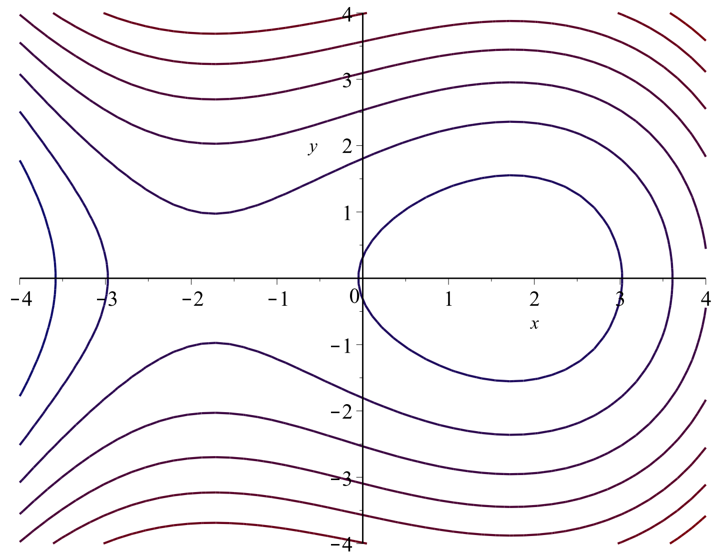
☐ $p(z)$ has precisely 9 roots, counting multiplicity.

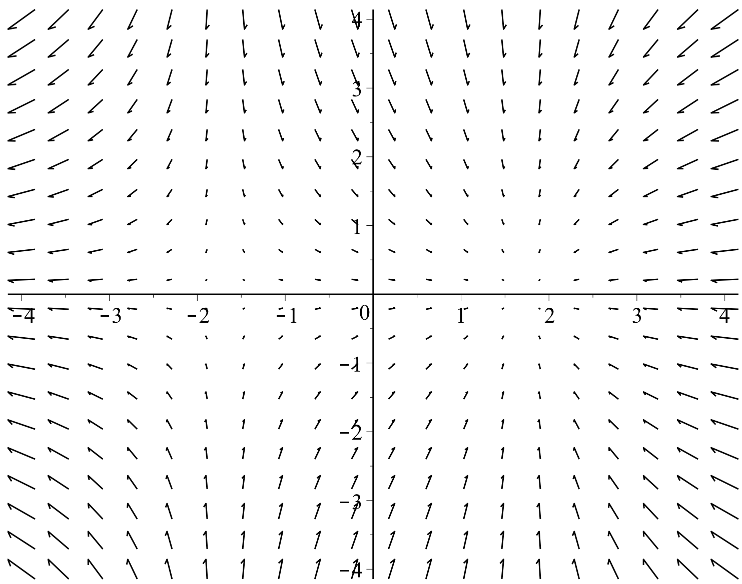
☐ If z_0 is a root of $p(z)$ then $\overline{z}_0$ is also a root of $p(z)$.

☐ $p(z)$ will always have 9 different roots.

Problem 12 (6%)

A differentiable function $f(x, y)$ is defined for $-4 \le x \le 4$ and $-4 \le y \le 4$. The two figures below show selected level curves and gradient vectors for the function. There are precisely two critical points with coordinates $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$. Determine the type of each critical point and mark your answer below.





a. At $(-\sqrt{3}, 0)$ the function f has a

- ☐ local maximum
- ☐ local minimum
- ☐ saddle point

b. At $(\sqrt{3}, 0)$ the function f has a

- ☐ local maximum
- ☐ local minimum
- ☐ saddel point