Exam in Calculus

Monday 8 June 2015

First Year at the Faculty of Engineering and Science
and
The Faculty of Medicine

The present exam set consists of 7 numbered pages with 12 problems.

It is allowed to use books, notes etc. It is **not allowed** to use **electronic devices**.

The listed percentages specify the weightings of the individual problems in the assessment of the exam.

The exam set has two independent parts.

- Part I contains "regular problems”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II consists of "multiple choice” problems. **The answers of Part II** must be given on **these sheets of paper**.

Remember to write your full name and student number on each page of your solutions. **Number each page** and write the **total number of pages** on the front page. Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Part I ("regular problems")

Problem 1 (8%)

A plane curve is given by

$$x = t^2,$$
$$y = 2t + 1,$$

where $t \in \mathbb{R}$.

(a) Find the curvature $\kappa(t)$ of the curve for $t \in \mathbb{R}$.

(b) Verify that $P = (0, 1)$ is a point on the curve and compute the curvature of the curve at this point.

Problem 2 (10%)

(a) Find the general solution of the differential equation

$$y'' - 2y' + 10y = 0.$$

(b) Find the general solution of the differential equation

$$y'' - 2y' + 10y = -13e^{3t}.$$

Problem 3 (7%)

Find the Taylor polynomial of degree 2 for the function

$$f(x) = 2 + x^2 + \arctan(x)$$

about the point $a = 0$.

Problem 4 (8%)

Consider the function

$$f(x,y) = \frac{x}{x^2+y^2}.$$

(a) Find the domain of f.

(b) Sketch the level curve with the equation $f(x,y) = 1$.

Problem 5 (12%)

A function is defined by

$$f(x,y) = x\sin(y) - y.$$

(a) Find the gradient vector $\nabla f(x,y)$.

(b) Compute the directional derivative of f at the point $P = (1, \frac{\pi}{2})$ in the direction of the unit vector

$$\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}.$$

(c) In which direction is the directional derivative of f at $P = (1, \frac{\pi}{2})$ maximal? (Specify a unit vector). What is the maximal value of the directional derivative of f at the point P?

Problem 6 (12%)

A plane region is given by

$$\mathscr{R} = \{(x,y): \quad 0 \le x \le 1, \quad x^2 - 1 \le y \le -x + 2\}.$$

(a) Sketch the region $\mathscr{R}$.

(b) Evaluate the double integral

$$\iint_{\mathscr{R}} 2xy\, dA.$$

Problem 7 (7%)

Find all complex roots of the polynomial

$$z^2 + (1-4i)z - 3 - 3i.$$

Problem 8 (8%)

The surface $\mathscr{F}$ in space is determined by the equation $F(x, y, z) = 0$ with

$$F(x, y, z) = xy^2 + e^y - xz^3.$$

(a) Verify that the point $P = (1, 0, 1)$ lies on the surface $\mathscr{F}$.

(b) Find an equation of the tangent plane to the surface $\mathscr{F}$ at the point $P = (1, 0, 1)$.

Part II ("multiple choice” problems)

Problem 9 (10%).

Answer the following 5 True/False problems:

a. The identity

$$\cos(\alpha+\beta)=\cos(\alpha)+\cos(\beta)$$

holds for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

☐ True ☐ False

b. For the function $f(t)=e^{(1-i)t}$, where $t \in \mathbb{R}$, one has

$$f'(t)=(1-i)e^{(1-i)t}.$$

☐ True ☐ False

c. Let $D=\{(x,y): x^2+y^2 \leq 1\}$ and let f be the function

$$f(x,y)=x^3y+\cos(2x)$$

defined on D. Then f attains an absolute maximum value at some point of D.

☐ True ☐ False

d. It holds true that

$$e^{6\pi i}=-1.$$

☐ True ☐ False

e. For all complex numbers $z \neq 0$ one has

$$\frac{z\overline{z}}{|z|^2}=1.$$

☐ True ☐ False

Remark. In problems 10 and 11 the evaluation of your answers will be performed using the following principle: Every false mark cancels a true mark.

Problem 10 (7%)

A solid body $\mathscr{T}$ covers precisely the region in space given by

$$\left\{(x,y,z): \quad (x,y)\in\mathscr{R}, \quad -x\le z\le y^2+3x\right\}$$

where

$$\mathscr{R}=\left\{(x,y): \quad -2\le y\le 2, \quad y^2\le x\le 4\right\}.$$

The density function of $\mathscr{T}$ is $\delta(x,y,z)=2x+z^2$. Mark all correct statements below. Remark: **Do not** evaluate the integrals.

- ☐ The mass of $\mathscr{T}$ can be calculated as follows: $m=\displaystyle\int_{y^2}^{4}\int_{-2}^{2}\int_{-x}^{y^2+3x}(2x+z^2)\,dz\,dy\,dx.$
- ☐ The mass of $\mathscr{T}$ can be calculated as follows: $m=\displaystyle\int_{-2}^{2}\int_{y^2}^{4}\int_{-x}^{y^2+3x}(2x+z^2)\,dz\,dx\,dy.$
- ☐ The mass of $\mathscr{T}$ can be calculated as follows: $m=\displaystyle\int_{0}^{4}\int_{-\sqrt{x}}^{\sqrt{x}}\int_{-x}^{y^2+3x}(2x+z^2)\,dz\,dy\,dx.$
- ☐ The volume of $\mathscr{T}$ can be calculated as follows: $V=\displaystyle\int_{y^2}^{4}\int_{-2}^{2}\int_{-x}^{2x+z^2}dz\,dy\,dx.$
- ☐ The volume of $\mathscr{T}$ can be calculated as follows: $V=\displaystyle\int_{-2}^{2}\int_{y^2}^{4}\int_{-x}^{y^2+3x}dz\,dx\,dy.$

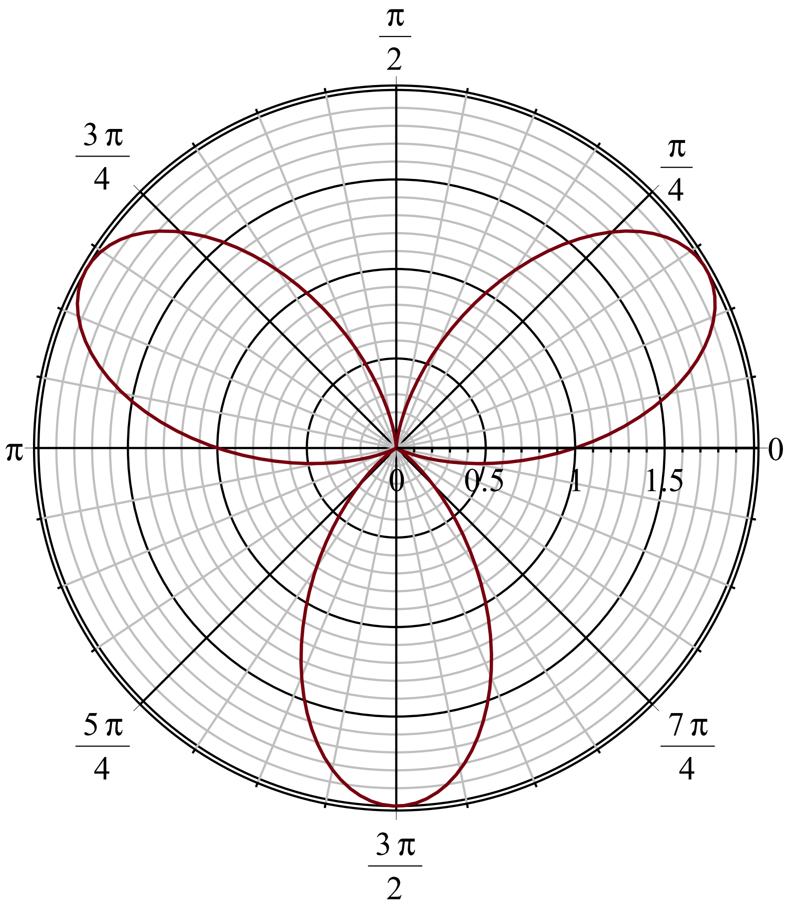
Problem 11 (6%)

Let $p(z)$ be a complex polynomial of degree 6. Suppose that $p(z)$ has real coefficients. Mark all correct statements below.

- ☐ $p(z)$ will always have 6 different complex roots.
- ☐ $p(z)$ will always have at least one real root.
- ☐ There will always exist a degree 2 polynomial $q_1(z)$ with real coefficients and a degree 4 polynomial $q_2(z)$ such that $p(z)=q_1(z)q_2(z)$.
- ☐ $p(z)$ will always have 6 complex roots, counting multiplicity.

Problem 12 (5%)

The figure below shows the graph of the function $r = f(\theta)$ displayed in polar coordinates.



Which one of the below expressions of f along with domain indication for θ corresponds to the figure above?

- ☐ $f(\theta) = 2 - \cos(3\theta),\ 0 \le \theta \le \frac{\pi}{3}$
- ☐ $f(\theta) = 1 + \cos(6\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 + \cos(3\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 + \sin(6\theta),\ 0 \le \theta \le \pi$
- ☐ $f(\theta) = 1 + \sin(3\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 2 - \sin(3\theta),\ 0 \le \theta \le 2\pi$