Exam in Calculus

Tuesday June 3 2014

First Year at The TEK-NAT Faculty and Health Faculty

The present exam consists of 7 numbered pages with a total of 12 exercises.

It is allowed to use books, notes, etc. It is **not** allowed to use electronic devices.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts.

- Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II is "multiple choice" exercises. The answers for Part II must be given on these sheets

Remember to write your full name (including middle names) and student number on each page of your answers. **Number each page and write the total number of pages on the front page of the answers.**

Good luck!

NAME: ________________________________

STUDENT NUMBER: ________________________________

Part I ("regular exercises")

Exercise 1 (8%)

A surface $\mathscr{F}$ in space is given by

$$z = 3 + x^3 + y^2, \qquad (x, y) \in \mathbb{R}^2.$$

(a) Verify that the point $P(1,2,8)$ belongs to $\mathscr{F}$.

(b) Determine an equation for the tangent plane to $\mathscr{F}$ through the point $P(1,2,8)$.

Exercise 2 (7%)

The Taylor polynomial of degree 3 for $f(x)$ about $a = 0$ is given by

$$P_3(x) = 2 + x + 4x^2 + 2x^3.$$

Determine $f(0)$ and $f'''(0)$.

Exercise 3 (12%)

(a) Find the general solution to the differential equation

$$y'' + 2y' + y = 0.$$

(b) A particular solution to the differential equation

$$y'' + 2y' + y = \cos(t)$$

is given by $y(t) = \frac{1}{2}\sin(t)$. Find the general solution to the differential equation

$$y'' + 2y' + y = 2 + \cos(t).$$

Exercise 4 (8%)

Compute all complex roots of the polynomial given by

$$p(z) = (z - 3i)(z^2 - 3iz - 2).$$

Exercise 5 (12%)

(a) The region $\mathscr{R}$ in the plane completely covers the triangle with corner points $(0,0)$, $(1,0)$ and $(0,3)$. Sketch $\mathscr{R}$.

(b) Calculate the double integral of $f(x,y) = x^2$ over the region $\mathscr{R}$.

Exercise 6 (12%)

Given the function

$$f(x,y,z) = 2z + \sin(z) + x^2 y^3, \qquad (x,y,z) \in \mathbb{R}^3.$$

(a) Determine the gradient vector $\nabla f(x,y,z)$.

(b) The function $g(x,y)$ is defined implicitly by $f(x,y,g(x,y)) = 1$. Determine $g(1,1)$ and $g_y(1,1)$.

Exercise 7 (5%)

Consider the function

$$f(x, y) = \log(x^2 + y^2).$$

(a) Determine the domain of f.

(b) Calculate the mixed partial derivative $f_{xy}(x, y)$.

Exercise 8 (8%)

Consider the plane curve given by

$$\mathbf{r}(t) = \cos(t)\,\mathbf{i} + 2\sin(t)\,\mathbf{j} = \begin{bmatrix} \cos(t) \\ 2\sin(t) \end{bmatrix}, \qquad t \in [0, 2\pi).$$

Determine $\mathbf{r}(\pi/4)$ and calculate the curvature of the curve at $\mathbf{r}(\pi/4)$.

Part II ("multiple choice" exercises)

Exercise 9 (10%).

Answer the following 6 true/false exercises:

a. We have the identity

$$\sin(2\alpha) = 2\cos(\alpha)\sin(\alpha)$$

for *all* $\alpha \in \mathbb{R}$.

☐ True ☐ False

b. A continuously differential function $f : \mathbb{R}^2 \to \mathbb{R}$ with a local minimum at $P \in \mathbb{R}^2$ satisfies $\nabla f(P) = \mathbf{0}$.

☐ True ☐ False

c. For any complex number $z \neq 0$ we have

$$\frac{z}{|z|} = \overline{z}.$$

☐ True ☐ False

d. The following identity holds true for the inverse sine function:

$$\sin(\arcsin(x)) = x, \qquad \text{for all } x \in (-\pi/2, \pi/2).$$

☐ True ☐ False

e. For $f(t) = e^{(1+2i)t}$, $t \in \mathbb{R}$, we have

$$f'(t) = (1+i)e^{(1+2i)t}, \qquad t \in \mathbb{R}.$$

☐ True ☐ False

f. We have the identity

$$\frac{d}{dx}\arctan(x^2) = \frac{2x}{1+x^4}$$

for *alle* $x \in \mathbb{R}$.

☐ True ☐ False

Notice. In exercises 10–12 the evaluation is performed using the following principle: Every false mark cancels one true mark. Therefore, one gains *nothing* from a "safe bet".

Exercise 10 (6%)

A body T covers the simple region in space given by

$$\{(x,y,z): \quad (x,z) \in \mathscr{R}, 0 \le y \le 3(1-x/2-z)\},$$

with

$$\mathscr{R} = \{(x,z): \quad 0 \le z \le 1, 0 \le x \le 2(1-z)\}.$$

The density of T is $\delta(x,y,z) = 1+z^3$. Mark all correct statements below (note: The value of the integral should *not* be computed).

- ☐ The mass of T can be computed as follows: $m = \int_0^1 \int_0^{2(1-z)} \int_0^{3(1-x/2-z)} (1+z^3)\, dydxdz$
- ☐ The volume of T can be computed as follows: $V = \int_0^1 \int_0^{2(1-z)} \int_0^{3(1-x/2-z)} dzdxdy$
- ☐ The mass of T can be computed as follows: $m = \int_0^{2(1-z)} \int_0^1 \int_0^{3(1-x/2-z)} (1+z^3)\, dxdydz$
- volume of T can be computed as follows: $V = \int_0^1 \int_0^{2(1-z)} \int_0^{3(1-x/2-z)} dydxdz$
- ☐ The mass of T can be computed as follows: $m = \int_0^1 \int_0^{2(1-z)} \int_0^{3(1-x/2-z)} (1+z^3)\, dydzdx$.

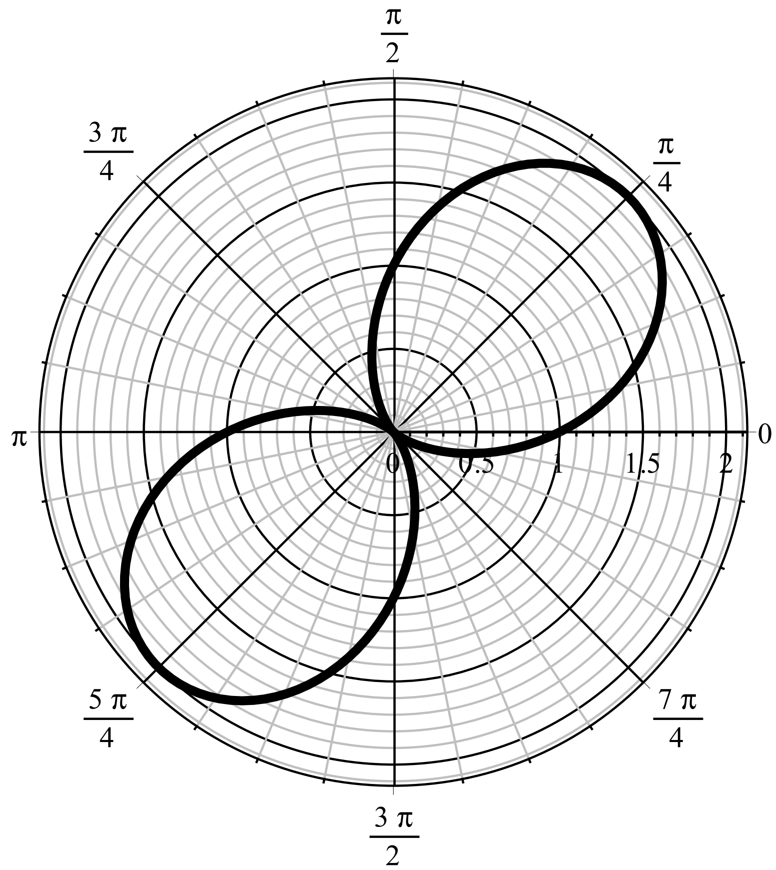
Exercise 11 (5%)

Let $p(z)$ be a complex polynomial of degree 5. Suppose that $p(z)$ has real coefficients. Mark all correct statements below.

- ☐ $p(z)$ has always 5 different complex roots
- ☐ $p(z)$ has always at least one real root
- ☐ $p(z)$ has 5 complex roots counting multiplicity
- ☐ $p(z)$ has at least one root on the imaginary axis.

Exercise 12 (7%)

The figure below shows the graph of the function $r = f(\theta)$ displayed in polar coordinates



Which of the expressions given below for f along with the domain for the θ correspond to the above figure.

- ☐ $f(\theta) = 2 + \cos(4\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 2 - \cos(2\theta),\ 0 \le \theta \le \pi$
- ☐ $f(\theta) = 1 + \cos(3\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 + \sin(2\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 2 + \sin(2\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = \cos(2\theta),\ 0 \le \theta \le \pi$.