Exam in Calculus

Tuesday June 11 2013

First Year at The TEK-NAT Faculty and Health Faculty

The present exam consists of 7 numbered pages with a total of 12 exercises.

It is allowed to use books, notes, etc. It is **not** allowed to use electronic devices.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts.

- Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II is "multiple choice" exercises. The answers for Part II must be given on these sheets

Remember to write your full name (including middle names) and student number on each page of your answers. **Number each page and write the total number of pages on the front page of the answers.**

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Part I ("regular exercises")

Exercise 1 (12%)

(a) Find the complete solution to the differential equation

$$y'' - 2y' + 5y = 0.$$

(b) Find the complete solution to the inhomogeneous differential equation

$$y'' - 2y' + 5y = 5t.$$

Exercise 2 (7%)

Determine the Taylor polynomial of degree 4 for

$$f(x) = \cos(2x),$$

around $a = 0$.

Exercise 3 (8%)

A surface $\mathscr{F}$ is given implicitly as the solution of the equation

$$F(x, y, z) = x^2 + 2y^2 + 3z^2 = 9.$$

(a) Verify that the point $P(2, -1, -1)$ is on $\mathscr{F}$.

(b) Determine an equation for the tangent plane to $\mathscr{F}$ through the point $P(2, -1, -1)$.

Exercise 4 (8%)

Find all complex solution to the binomial equation

$$z^5 + i = 0.$$

The solutions may be presented in polar form.

Exercise 5 (16%)

(a) The region $\mathscr{R}$ in the xy-plane is bounded by the curves $y = x^2$ and $y = 2 - x^2$. Sketch $\mathscr{R}$.

(b) Compute the integral of $f(x, y) = x^2$ over the planar region $\mathscr{R}$.

(c) A body T with density $\delta(x, y, z) = z$ covers exactly the region in space given by

$$\{(x, y, z) : (x, y) \in \mathscr{R}, 0 \le z \le x^2\}.$$

Determine the mass of T.

Exercise 6 (8%)

Consider the function

$$f(x, y, z) = \sin(\pi x) + xy^2 z^5, \qquad (x, y, z) \in \mathbb{R}^3.$$

(a) Determine the gradient $\nabla f(x, y, z)$.

(b) Find the directional derivatives of f in the point $P(-1/2, 1, -1)$ in the direction given by

$$\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Exercise 7 (5%)

Consider the function

$$f(x, y) = \arcsin(x^2 + y^2).$$

(a) Determine the domain of f.

(b) Determine the partial derivatives of $f_x(x, y)$ and $f_y(x, y)$.

Exercise 8 (8%)

Consider the parametric curve given by

$$\mathbf{r}(t) = \cos(2t)\,\mathbf{i} + \sin(2t)\,\mathbf{j} + 2\sqrt{3}\,t\,\mathbf{k} = \begin{bmatrix} \cos(2t) \\ \sin(2t) \\ 2\sqrt{3}\,t \end{bmatrix}, \qquad t \in \mathbb{R}.$$

Compute the arc length of the curve from $t = 0$ to $t = 2$.

Part II (“multiple choice” exercises)

Notice. In exercises 9 and 10 the evaluation is performed using the following princip-le: Every false mark cancels one true mark. Therefore, one gains *nothing* from a “safe bet”.

Exercise 9 (6%)

A body T covers the area in space that is given by

$$\{(x, y, z): \quad (x, y) \in R, x^2 \le z \le 2x^2 + y^2\},$$

where

$$R = \{(x, y): \quad -1 \le y \le 2, 1 - y^2 \le x \le 1 + y^2\}.$$

The density of T is $\delta(x, y, z) = z$. Mark all correct statements below (note: The value of the integral should *not* be computed).

☐ The mass of T can be computed as follows: $m = \int_{-1}^{2} \int_{1-y^2}^{1+y^2} \int_{x^2}^{2x^2+y^2} x^2 z \, dz dx dy$

☐ The volume of T can be computed as follows: $V = \int_{-1}^{2} \int_{1-y^2}^{1+y^2} \int_{x^2}^{2x^2+y^2} dz dx dy$

☐ The mass of T can be computed as follows: $m = \int_{-1}^{2} \int_{1-y^2}^{1+y^2} \int_{x^2}^{2x^2+y^2} z \, dy dz dx$

☐ The volume of T can be computed as follows: $V = \int_{-1}^{2} \int_{1-y^2}^{1+y^2} \int_{x^2}^{2x^2+y^2} dz dy dx$

☐ The mass of T can be computed as follows: $m = \int_{-1}^{2} \int_{1-y^2}^{1+y^2} \int_{x^2}^{2x^2+y^2} z \, dz dx dy$

Exercise 10 (5%)

Let $f(x, y)$ be a continuous real function defined on a region R in the plane. R consists of the points on and inside a simple closed curve C.

Mark all correct statements below.

☐ f always has a global maximum on R.

☐ If f does *not* have a global extremum on C, then f has a global extremum *inside* C.

☐ If f has a horizontal tangent plane at the point $\mathbf{a} \in R$, then f has a local extremum in $\mathbf{a}$.

☐ If f has a local extremum in $\mathbf{a} \in R$, then all directional derivatives for f exists in $\mathbf{a}$.

Exercise 11 (10%).

Answer the following 6 true/false exercises

a. A differentiable curve $\mathbf{r} : (a, b) \to \mathbb{R}^3$ fulfils that $\mathbf{r}(t) \cdot \mathbf{r}(t) = 1$ for $t \in (a, b)$. Then

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0, \quad t \in (a, b).$$

☐ True ☐ False

b. For a complex number $w \neq 0$ we have that

$$\frac{1}{w} = \frac{\overline{w}}{|w|^2}.$$

☐ True ☐ False

c. The following equation hold for the inverse sine function:

$$\arcsin(x) = \frac{1}{\sin(x)}, \qquad x \in (-1, 1).$$

☐ True ☐ False

d. A function $f : \mathbb{R} \to \mathbb{R}$ is called surjective (onto) if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$, where $x_1, x_2 \in \mathbb{R}$.

☐ True ☐ False

e. We have that

$$\tan\big(\arctan(x)\big) = x$$

for *all* $x \in \mathbb{R}$.

☐ True ☐ False

f. We have that

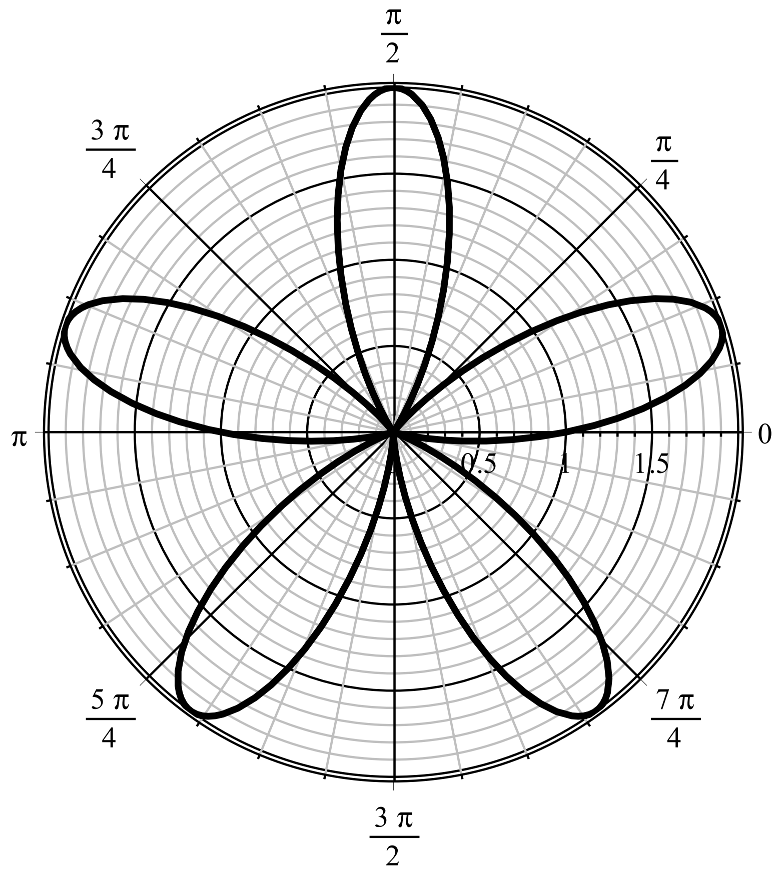
$$\sin^2(x) = 1 - \cos(2x)$$

for *all* $x \in \mathbb{R}$.

☐ True ☐ False

Exercise 12 (7%)

The figure below shows the graph of the function $r = f(\theta)$ displayed in polar coordinates



Which of the below expressions for f along with the domain for the θ gives the above figure.

- ☐ $f(\theta) = 1 + \cos(4\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 - \cos(7\theta),\ 0 \le \theta \le \pi$
- ☐ $f(\theta) = 1 + \cos(5\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 + \sin(5\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 2 + \sin(5\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = \cos(5\theta),\ 0 \le \theta \le \pi.$