Exam in Calculus

Monday June 4th 2012

First Year at The TEK-NAT Faculty and Health Faculty

The present exam consists of 7 numbered pages with a total of 12 exercises.

It is allowed to use books, notes, etc. It is **not** allowed to use electronic devices.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts.

- Part I contains “regular exercises”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II is “multiple choice” exercises. The answers for Part II must be given on these sheets

Remember to write your full name (including middle names), student number together with the course number on each side of your answers. **Number each page and write the total number of pages on the front page of the answers.**

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

COURSE NUMBER: ______________________________

Part I ("regular exercises")

Exercise 1 (8%)

A surface $\mathscr{F}$ is given implicitly as the solution of the equation

$$F(x,y,z) = x^3 + y^2 + z^4 - 3 = 0.$$

(a) Determine the gradient vector $\nabla F(x,y,z)$.

(b) Determine an equation for the tangent plane to $\mathscr{F}$ through the point $(1,-1,-1)$.

Exercise 2 (16%)

(a) Find the uniquely determined solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0,$$

that fulfills

$$y(0) = 0 \quad \text{and} \quad y'(0) = 3.$$

(b) Find a particular solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 1.$$

(c) We are informed that

$$y_p(x) = \cos x - 3\sin x$$

is a particular solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 10\sin x.$$

Find the complete solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 1 + 30\sin x.$$

Exercise 3 (8%)

Solve the complex 2. degree equation

$$z^2 + (1+i)z + 2 - i = 0.$$

Exercise 4 (9%)

(a) The area $\mathscr{R}$ in xy plane covers exactly the triangle with the corners $(-1,1)$, $(0,0)$ and $(1,1)$. Sketch $\mathscr{R}$.

(b) Find the double integral of $f(x,y) = 3y$ over $\mathscr{R}$.

Exercise 5 (7%)

(a) Write down the Taylor polynomial of degree 2 for

$$f(x) = 1 + 4x + 8x^2 + 12x^3,$$

around $a = 0$.

(b) We are told that $g(x)$ is 6 times differentiable and has the following Taylor polynomial of degree 2

$$P_2(x) = 1 + 2(x-1) + 4(x-1)^2$$

around $a = 1$. Determine $g''(1)$.

Exercise 6 (10%)

A thin plate covers exactly the area

$$\mathscr{R} = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4,\ x \geq 0\}$$

in the xy plane. The density of the plate is $\delta(x, y) = x$.

(a) Determine the mass of the plate.

Let $(\bar{x}, \bar{y})$ denote the centroid of the plate. A calculation shows that $\bar{x} = \frac{45\pi}{112}$.

(b) Compute $\bar{y}$.

Exercise 7 (5%)

Consider the function

$$f(x, y) = \arctan(x^2 + y^3).$$

(a) Determine the domain of f.

(b) Determine the partial derivatives $f_x(x, y)$ and $f_y(x, y)$.

Exercise 8 (5%)

Consider the planar curve

$$\begin{aligned} x &= t \\ y &= \sin t, \end{aligned}$$

where $t \in \mathbb{R}$. Compute the curvature $\kappa(t)$ of the curve for $t = \frac{3\pi}{4}$.

Part II (”multiple choice” exercises)

Notice. In exercises 9 and 10 the evaluation is performed using the following principle: Every false mark cancels one true mark. Therefore, one gains *nothing* from a “safe bet”.

Exercise 9 (6%)

A body T covers the area in space that is given in spherical coordinates by

$$\{(\rho,\phi,\theta): \quad 0 \le \phi \le \pi/6, \quad 0 \le \rho \le \pi/8\}.$$

The density of T is $\delta(x,y,z) = y^2$. Which of the following 4 iterated integrals can be used to evaluate the mass of T (note: The value of $\bar{x}$ should *not* be computed)

☐ $\int_0^{\pi} \int_0^{\pi/6} \int_0^{\pi/8} \rho^3 \sin\theta \sin^2\phi \, d\rho \, d\phi \, d\theta.$

☐ $\int_0^{2\pi} \int_0^{\pi/6} \int_0^{\pi/8} \rho^3 \sin\theta \sin^2\phi \, d\rho \, d\phi \, d\theta.$

☐ $\int_0^{2\pi} \int_0^{\pi/6} \int_0^{\pi/8} \rho^4 \sin^2\theta \sin^3\phi \, d\rho \, d\phi \, d\theta.$

☐ $\int_0^{2\pi} \int_0^{\pi/6} \int_0^{\pi/8} \rho^4 \sin^3\theta \sin^2\phi \, d\rho \, d\phi \, d\theta.$

Exercise 10 (5%)

Consider two complex numbers $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Mark all correct statements below

☐ $c_1 c_2 = a_1 a_2 + b_1 b_2 + i(a_1 b_1 + b_1 a_2).$

☐ $c_1 = \overline{c_1}$, if and only if $a_1 = 0$.

☐ $\frac{c_1}{c_2}$ is defined if $a_2^2 + b_2^2 > 0$.

☐ The equation with z given by $z^2 + c_1 = z - c_2$ has exactly two complex solutions counted with multiplicity.

Exercise 11 (14%).

Answer the following 5 true/false exercises:

a. Consider a continuous function f defined on an area $\mathscr{R}$ in the plane, where $\mathscr{R}$ is given in polar coordinates by

$$\mathscr{R} = \{(r,\theta)_{\text{pol}} : r_1(\theta) \le r \le r_2(\theta),\ \alpha \le \theta \le \beta\},$$

where r_1 and r_2 are continuous functions and $\alpha \le \beta$ are constants. The corresponding double integral can be evaluated as the following iterated integral.

$$\iint_{\mathscr{R}} f(x,y)\,dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta.$$

☐ True ☐ False

b. A continuous function defined on an area $\mathscr{R}$ in the xy plane, where $\mathscr{R}$ consists of the points on and inside a simple closed curve always has a global minimum on $\mathscr{R}$.

☐ True ☐ False

c. Consider a differentiable function $f(x,y)$ defined on $\mathbb{R}^2$. If all directional derivatives of f in the point $(0,0)$ are 0 then f has a local maximum in $(0,0)$.

☐ True ☐ False

d. A function $f : \mathbb{R} \to \mathbb{R}$ is called one-to-one if $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{R}$.

☐ True ☐ False

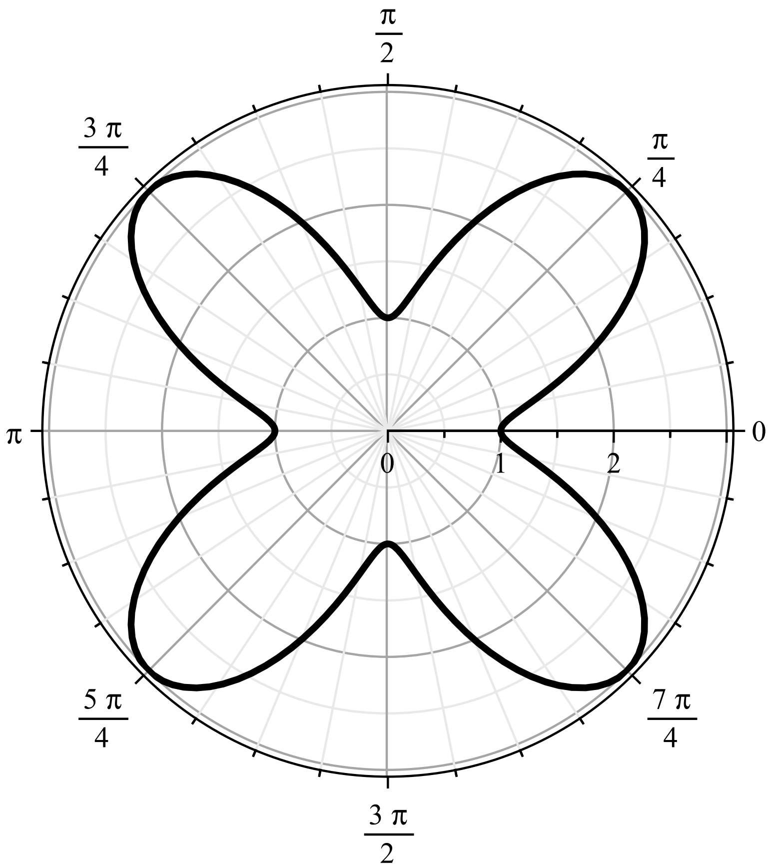
e. We have that

$$\frac{d}{dx}\big(\arcsin x + \arccos x\big) = 0, \qquad x \in (-1,1).$$

☐ True ☐ False

Exercise 12 (7%)

The figure below shows the graph of the function $r = f(\theta)$ displayed in polar coordinates



Which of the below expressions for f along with the domain for the θ gives the above figure.

- ☐ $f(\theta) = 1 + \cos(4\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 2 - \cos(2\theta),\ 0 \le \theta \le \pi$
- ☐ $f(\theta) = 2 + \sin(4\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 2 - \cos(4\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 + 2\cos(2\theta),\ 0 \le \theta \le 2\pi$
- ☐ $f(\theta) = 1 + \cos(2\theta),\ 0 \le \theta \le \pi.$