Exam in Calculus
Wednesday June 1st 2011

First Year at The TEK-NAT Faculty and Health Faculty

The present exam consists of 7 numbered pages with a total of 12 exercises.

It is allowed to use books, notes, etc. It is **not** allowed to use electronic devices.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts.

- Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II is "multiple choice" exercises. The answers for Part II must be given on these sheets

Remember to write your full name (including middle names), student number together with the course number on each side of your answers. **Number each page and write the total number of pages on the front page of the answers.**

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

COURSE NUMBER: ______________________________

Part I (”regular exercises”)

Exercise 1 (12%)

(a) Find the uniquely determined solution to the initial value problem

$$\begin{cases} \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0 \\ y(0) = 0 \quad \text{og} \quad y'(0) = 1. \end{cases}$$

(b) Find the complete solution to the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x + e^x.$$

Exercise 2 (7%)

Consider the function

$$f(x,y) = \cos(x^3 - y^2).$$

(a) Determine the partial derivatives $f_x(x,y)$ and $f_y(x,y)$.

(b) Determine the mixed 2. order partial derivative $f_{xy}(x,y)$.

Exercise 3 (10%)

Compute the 3. order Taylor polynomial for

$$f(x) = \arccos(2x) + x^2 + x,$$

around $a = 0$.

Exercise 4 (8%)

Compute the double integral

$$\iint_{\mathcal{R}} x \, dA,$$

where $\mathcal{R}$ is bounded in the xy plane by the curves $y = x^2$ and $x = y^2$.

Exercise 5 (7%)

Solve the second degree equation

$$z^2 - (2 - i)z = -2 - 2i.$$

Exercise 6 (8%)

Consider the surface F that is implicitly defined as the solution of the equation

$$x + 2y^2 + 3z^3 = 17.$$

Find the tangent plane of F in the point $P(6, 2, 1)$.

Exercise 7 (8%)

Consider the function

$$f(x, y, z) = \sin[\pi(x + y)] + z^3.$$

(a) Determine the gradient $\nabla f(x, y, z)$.

(b) Determine the directional derivative f in the point $P(1, 1, 1)$ in the direction given by $\mathbf{v} = \mathbf{i} - \mathbf{j} - \mathbf{k}$.

Exercise 8 (10%)

The body T in space is bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $z = 2$. The density of T is given by $\delta(x, y, z) = z$.

(a) Determine the mass of T.

(b) The centroid of T is written $(\bar{x}, \bar{y}, \bar{z})$. Considerations involving symmetry show that $\bar{x} = \bar{y} = 0$. Determine $\bar{z}$.

Part II ("multiple choice" exercises)

Notice. In exercises 9 and 10 the evaluation is performed using the following principle: Every false mark cancels one true mark. Therefore, one gains *nothing* from a "safe bet".

Exercise 9 (6%)

A body T covers the area in space that is given in spherical coordinates by

$$\{(\rho, \phi, \theta) : 0 \leq \phi \leq \pi/3, 0 \leq \rho \leq \pi/4\}.$$

The density of T is $\delta(x, y, z) = 1$, T's mass is named m_T and T's centroid is named $(\bar{x}, \bar{y}, \bar{z})$.

Which of the following 4 iterated integrals can be used to evaluate $\bar{x}$ (note: The value of $\bar{x}$ should *not* be computed)

- ☐ $\dfrac{1}{m_T}\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^{\pi/4} \rho^2 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta.$
- ☐ $\dfrac{1}{m_T}\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^{\pi/4} \rho^3 \cos\theta \sin^2\phi \, d\rho \, d\phi \, d\theta.$
- ☐ $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^{\pi/4} \rho^3 \cos\theta \sin^2\phi \, d\rho \, d\phi \, d\theta.$
- ☐ $\dfrac{1}{m_T}\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\pi/3} \rho^2 \cos\phi \, d\rho \, d\phi \, d\theta.$
- ☐ $\dfrac{1}{m_T}\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\pi/3} \rho^3 \cos\theta \sin^2\phi \, d\rho \, d\phi \, d\theta.$

Exercise 10 (4%)

Consider a complex polynomial $p(z)$ of degree 4. Mark all correct statements below

- ☐ $p(z)$ always has at least one real root
- ☐ $p(z)$ has exactly 4 different complex roots
- ☐ $p(z)$ has exactly 4 roots counted with multiplicity
- ☐ Der exists a $z_0 \in \mathbb{C}$, such that $p(z) = (z - z_0)q(z)$ where $q(z)$ is a complex polynomial of degree 3

Exercise 11 (12%).

Answer the following 5 true/false exercises:

a. Every continuous function defined on an area $\mathcal{R}$ in the xy plane always has a global maximum on $\mathcal{R}$.

☐ True ☐ False

b. The double integral over the area

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d,\ x_1(y) \leq x \leq x_2(y)\},$$

where c and d are constants and x_1 and x_2 are continuous functions can be evaluated as the following iterated integral

$$\iint_{\mathcal{R}} f(x, y)\, dA = \int_c^d \int_{x_1(y)}^{x_2(y)} f(x, y)\, dx\, dy.$$

☐ True ☐ False

c. For every choice of the constant $a \in \mathbb{R}$, the initial value problem

$$\begin{cases} \frac{d^2y}{dx^2} + a\frac{dy}{dx} + 12y = \cos^3(x) \\ y(5) = 2,\ y'(5) = -6, \end{cases} \qquad x \in \mathbb{R},$$

has exactly one solution.

☐ True ☐ False

d. There exists a continuous function $f(x, y)$, defined on $\mathbb{R}^2$, where all directional derivatives exists in $(0, 0)$, but f is not differentiable in $(0, 0)$.

☐ True ☐ False

e. There exists a complex polynomial of degree 9 with real coefficients that cannot be factorised as a product of linear and quadratic real factors.

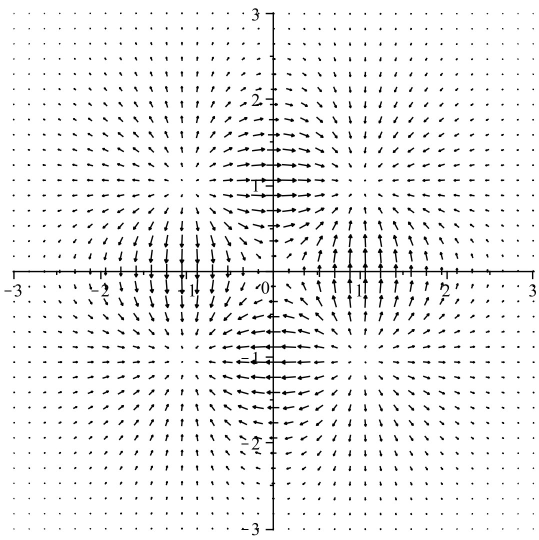
☐ True ☐ False

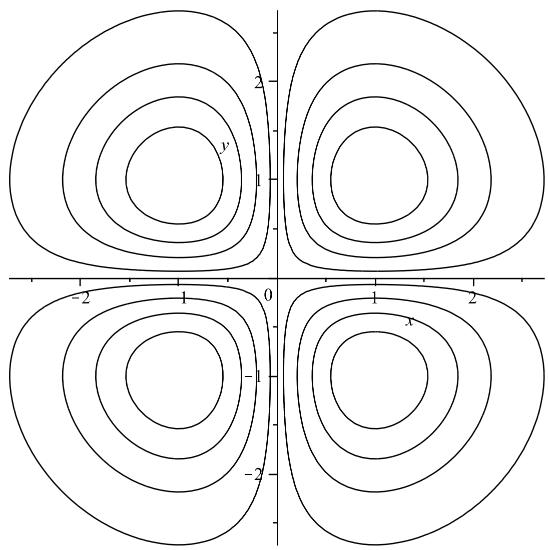
Exercise 12 (8%)

A differentiable function $f(x, y)$ is defined on the square

$$R = \{(x, y) : -3 \le x, y \le 3\}.$$

The two figures below show certain gradient vectors and level curves, respectively, of the function on R. The function has four critical points in R with coordinates $\pm(1, 1)$ and $\pm(1, -1)$. Use the figures to determine the nature of each of the critical points and mark the answer below.





(a) In the point $(1, 1)$ f has a:

- ☐ local maximum
- ☐ local minimum
- ☐ saddle point.

(b) In the point $(-1, 1)$ f has a:

- ☐ local maximum
- ☐ local minimum
- ☐ saddle point.

(c) In the point $(-1, -1)$ f has a:

- ☐ local maximum
- ☐ local minimum
- ☐ saddle point.

(d) In the point $(1, -1)$ f has a:

- ☐ local maximum
- ☐ local minimum
- ☐ saddle point.