Retake exam in Calculus

19. august 2019

Exercise 1 (6 point)

A function is defined by

$$f(x, y, z) = \sin(x^2 + y^2 + z^2)$$

for real variables x, y and z.

(a) (3 point) The domain of definition of f consists of all the points (x, y, z) which satisfy

- ☐ $z = 0$
- ☐ $x^2 + y^2 + z^2 > 0$
- ☐ $x^2 + y^2 + z^2 < 0$
- ☑ all points are allowed
- ☐ $xyz \neq 0$
- ☐ none of the others

(b) (3 point) Which points (x, y, z) belong to the level surface determined by $f(x, y, z) = 2$?

- ☐ A sphere given by $x^2 + y^2 + z^2 = \sin^{-1}(2)$
- ☐ The xy-plane
- ☑ There are no such points
- ☐ A plane parallel with the xy-plane given by $z = \sin^{-1}(2)$
- ☐ none of the others

Exercise 2 (6 point)

A parametric curve in space is given by

$$\mathbf{r}(t) = \langle \sin(t),\ \ \cos(t),\ \ t \rangle$$

where the parameter t is an arbitrary real number.

(a) (2 point) What is the velocity vector of the curve?

- ☐ $\langle \sin(t), \cos(t), 1 \rangle$
- ☐ $\langle \cos(t), \sin(t), 1 \rangle$
- ☑ $\langle \cos(t), -\sin(t), 1 \rangle$
- ☐ $\langle \sin(t^2/2), \cos(t^2/2), t^2/2 \rangle$
- ☐ $\langle \sin(t^2/2), -\cos(t^2/2), t^2/2 \rangle$
- ☐ none of the others

(b) (2 point) Which of the following vectors is the acceleration vector for $t = 2\pi$?

☑ $\langle 0,\ -1,\ 0\rangle$ ☐ $\langle 1,\ 0,\ 0\rangle$ ☐ $\langle 0,\ 1,\ 1/2\rangle$
☐ $\langle 0,\ 1,\ 0\rangle$ ☐ $\langle 1,\ 1,\ 1\rangle$ ☐ none of the others

(c) (1 point) What is the speed?

☐ $\sqrt{\sin(t)+\cos(t)+t}$ ☐ 2 ☐ $\sqrt{t+1}$
☐ $\sqrt{1+t^2}$ ☑ $\sqrt{2}$ ☐ none of the others

(d) (1 point) What is the length of the curve from $t=0$ to $t=2$?

☐ 0 ☑ $2\sqrt{2}$ ☐ $1/2$
☐ $\sqrt{2}$ ☐ $4\sqrt{2}$ ☐ none of the others

Exercise 3 (6 point)

Three complex numbers are given by

$$z_1 = 1+2i, \quad z_2 = 4-2i \quad \text{og} \quad z_3 = i^{100}.$$

(a) (2 point) What is $z_1 + z_2$ in polar form?

☐ 1 ☐ $5e^{-i\pi/2}$ ☑ 5
☐ $5e^{i\pi/2}$ ☐ $\sqrt{5}$ ☐ none of the others

(b) (2 point) What is $\frac{2z_1}{z_2}$ in the standard form $a+ib$?

☐ $1+i$ ☐ $-1-i$ ☐ 5
☑ i ☐ $-i$ ☐ none of the others

(c) (2 point) What is the principal argument of z_3?

☑ 0 ☐ $\pi/2$ ☐ π
☐ $\pi/4$ ☐ $3\pi/4$ ☐ none of the others

Hint to (c): The principal argument is a polar angle which belongs to the interval $]-\pi,\pi]$.

Exercise 4 (10 point)

(a) (5 point) A homogeneous second order differential equation is given by

$$y'' = -4y.$$

Below there are given a few functions where c_1 and c_2 are arbitrary real constants. Mark which expression represents the general solution to the differential equation.

- ☐ $y(t) = c_1 e^{-2t} + c_2 e^{2t}$
- ☐ $y(t) = c_1 e^{t} + c_2 t$
- ☐ $y(t) = c_1 \cos(t) + c_2 \sin(t)$
- ☑ $y(t) = c_1 \sin(2t) + c_2 \cos(2t)$
- ☐ $y(t) = (c_1 + c_2 t)e^{-2t}$
- ☐ $y(t) = (c_1 + c_2 t)e^{2t}$
- ☐ $y(t) = c_1 + c_2$
- ☐ none of the others

(b) (2 point) Which function $x_p(t)$ is a particular solution to the inhomogeneous differential equation

$$x''(t) = -4x(t) + 5e^t$$

among the following expressions:

- ☐ $x_p(t) = e^{2t}$
- ☐ $x_p(t) = e^{-t}$
- ☐ $x_p(t) = e^{-2t}$
- ☐ $x_p(t) = te^{t}$
- ☑ $x_p(t) = e^{t}$
- ☐ none of the others

(c) (3 point) Mark the solution $x(t)$ to the initial value problem

$$x''(t) = -4x(t) + 5e^t, \quad x(0) = 0, \quad x'(0) = 0,$$

among the following expressions:

- ☐ $x(t) = \sin(t) - t\cos(t)$
- ☑ $x(t) = -\sin(2t)/2 - \cos(2t) + e^t$
- ☐ $x(t) = -\sin(t) + te^t$
- ☐ $x(t) = t - te^{2t}$
- ☐ $x(t) = \sin(2t)/2 + \cos(2t) - e^t$
- ☐ none of the others

Exercise 5 (8 point)

Mark if the following statements are true or false:

(a) (2 point) $(\sqrt{2}+\sqrt{3})^2 = 5 + 2\sqrt{6}$.

☑ True ☐ False

(b) (2 point) $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^2 = i$.

☑ True ☐ False

(c) (2 point) If $f(x) = \sin(x)$ and $g(t) = \cos(t)$ then $h(t) = f(g(t))$ is differentiable and $h'(t) = -\sin(t)\cos(\cos(t))$.

☑ True ☐ False

(d) (2 point) $e^{-\ln(x)} = \frac{1}{x}$.

☑ True ☐ False

Opgave 6 (7 point)

A domain $\mathcal{R}$ in the plane can be represented with the help of the inequality $x^2 + y^2 \leq 1$.

(a) (1 point) Which of the following curves describes the boundary of $\mathcal{R}$?

☐ a circle centred at $(0,1)$ and with radius 1

☑ a circle centred at $(0,0)$ and with radius 1

☐ a triangle with corners at the origin, $(0,1)$ and $(1,0)$

☐ a square with side length 1 and center at the origin

☐ none of the others

(b) (2 point) Which of the following inequalities characterize all the points of $\mathcal{R}$, where $(x,y) = (r\cos(\theta), r\sin(\theta))$?

☐ $r \geq 1, \quad 0 \leq \theta \leq \pi$

☑ $0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$

☐ $0 \leq r \leq \cos(\theta), \quad 0 \leq \theta \leq \pi/2$

☐ $r \geq 1$

☐ $0 \leq r \leq \sin(\theta), \quad 0 \leq \theta \leq \pi$

☐ none of the others

(c) (4 point) The domain $\mathcal{R}$ models a plate with mass density $\delta(x,y) = x^2 + y^2$. Which of the following expressions give the mass of the plate?

- [ ] $2\pi \int_0^1 r^2 dr$
- [x] $2\pi \int_0^1 r^3 dr$
- [ ] $\pi \int_0^1 r^3 dr$
- [ ] $\int_0^{\pi} \int_0^{\sin(\theta)} r^3 dr\, d\theta$
- [ ] $\int_0^1 r^2 dr$
- [ ] none of the others

Opgave 7 (8 point)

A domain $\mathcal{R}$ in the plane consists of all the points with coordinates (x, y) which satisfy the inequality $x^2 + y^2 \leq 1$. The function f is defined on $\mathcal{R}$ and given by $f(x, y) = x + y$.

(a) (4 point) Which of the following points are inner critical points for f?

- [ ] $\langle 0, -1\rangle$
- [ ] $\langle 1, 0\rangle$
- [ ] $\langle 0, 0\rangle$
- [ ] $\langle -1, 0\rangle$
- [x] There are no inner critical points
- [ ] none of the others

(b) (2 point) Mark whether the following statement is true or false:

The function f evaluated on the boundary of $\mathcal{R}$ takes the same values as the function $g(\theta) = \cos(\theta) + \sin(\theta)$, where $\theta \in [0, 2\pi]$.

- [x] True
- [ ] False

(c) (2 point) What is the maximal value of f?

- [ ] 1
- [x] $\sqrt{2}$
- [ ] 2
- [ ] $2\sqrt{2}$
- [ ] 3
- [ ] none of the others

Exercise 8 (12 point)

A surface $\mathcal{F}$ in space is determined by the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = x^4 + y^4 - 2z^2$$

(a) (3 point) Which of the following expressions gives the gradient vector ∇F?

- [ ] $\langle 4x, 4y, -4z\rangle$
- [ ] $\langle 4x^2, 4y^2, -4z^2\rangle$
- [x] $\langle 4x^3, 4y^3, -4z\rangle$
- [ ] $\langle x^3, y^3, -2\rangle$
- [ ] $\langle 0, 0, 0\rangle$
- [ ] none of the others

(b) (4 point) Which of the following equations represent the tangent plane to $\mathcal{F}$ at the point $P = (1, 1, 1)$?

- [ ] $0 = x + y + z$
- [x] $x + y - z = 1$
- [ ] $z = x + y$
- [ ] $z = y + 2x - 2$
- [ ] $z = 2x - y$
- [ ] none of the others

(c) (5 point) From the equation $F(x, y, z) = 0$, what is the partial derivative $\partial z/\partial x$ at the point P?

☐ -1	☐ -2	☐ 2
☐ 0	☑ 1	☐ none of the others

Exercise 9 (12 point)

A function is given by

$$f(x, y) = \sin(2x + y),$$

where $x \geq 0$ and $y \geq 0$.

(a) (2 point) Mark whether the following statement is true or false: $f(x, y)$ can never be equal to zero.

☐ True	☑ False

(b) (2 point) Mark whether the following statement is true or false: $f(x, 0)$ is an increasing function of x.

☐ True	☑ False

(c) (4 point) What is the directional derivative $D_{\mathbf{u}} f(P)$ at the point $P = (0, 0)$ in the direction given by the unit vector $\mathbf{u} = \langle 0, 1 \rangle$?

☐ 0	☐ 3	☐ 4
☑ 1	☐ 2	☐ none of the others

(d) (4 point) Which of the following unit vectors point in the direction in which f grows fastest at the point P (the direction $\mathbf{v}$ for which $D_{\mathbf{v}} f(P)$ is largest)?

☐ $\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$	☐ $\langle \frac{2\sqrt{5}}{5}, -\frac{1}{\sqrt{5}} \rangle$	☐ $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$
☐ $\langle 1, 0 \rangle$	☑ $\langle \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \rangle$	☐ $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$
☐ $\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$	☐ $\langle 0, 1 \rangle$	☐ none of the others

Exercise 10 (9 point)

A function is given by

$$f(x) = \cos(x^2)$$

for all real numbers x.

(a) (5 point) Mark the correct expression for $f''(x)$ (i.e. f twice differentiated)

☐ $-4\cos(x^2)$
☐ $-x^4\cos(x^2)$
☑ $-2\sin(x^2) - 4x^2\cos(x^2)$
☐ $2\cos(2x)$
☐ $2\sin(x^2) - x^2\cos(x^2)$
☐ none of the others

(b) (4 point) Which of the following expressions represents the second order Taylor polynomial for f with the expansion point $x = 0$?

☐ $1 + x + x^2$
☐ $1 + x + x^2/2$
☐ $1 - x + x^2/2$
☑ 1
☐ $2x + x^2$
☐ none of the others

Exercise 11 (11 point)

A curve in the plane is given by

$$x(t) = t^2,$$
$$y(t) = \sin(t^3)$$

for all real numbers t.

(a) (2 point) For which value of the parameter t does the curve go through the origin?

☐ π
☐ 2π
☐ 3π
☐ 4π
☑ 0
☐ none of the others

(b) (4 point) What is the value of the speed when $t = 0$?

☑ 0
☐ 1
☐ $\sqrt{2}$
☐ $\sqrt{3}$
☐ $\sqrt{5}$
☐ none of the others

(c) (5 point) What is the acceleration vector at the origin?

☐ $\langle 0, 0\rangle$
☐ $\langle 1, 0\rangle$
☐ $\langle 0, 1\rangle$
☐ $\langle 1, 1\rangle$
☑ $\langle 2, 0\rangle$
☐ none of the others

Exercise 12 (5 point)

Consider the following initial value problem

$$y'(x) = (x+1)\,y(x), \quad y(0) = 1.$$

(a) (3 point) Assume that y solves the above equation and define

$$f(x) = \ln(y(x)).$$

Which initial value problem solves f?

- ☐ $f'(x) = x + 1, \quad f(0) = 1$
- ☐ $f'(x) = -x - 1, \quad f(0) = 1$
- ☑ $f'(x) = x + 1, \quad f(0) = 0$
- ☐ $f'(x) = e^x, \quad f(0) = 0$
- ☐ $f'(x) = 1/(x+1), \quad f(0) = 0$
- ☐ none of the others

(b) (2 point) What is $y(x)$?

- ☐ $1 + x$
- ☐ $1/(x+1)$
- ☐ $\ln(1 + x + x^2/2)$
- ☐ $e^x - 1$
- ☑ $e^{x+x^2/2}$
- ☐ none of the others