

Reexam in Calculus

First Year at the Faculty of Engineering and Science
and the Technical Faculty of IT and Design

26. Februar 2020

The present exam set consists of 8 numbered pages with 12 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use books, notes, etc. It is **not** allowed to use electronic devices.

Problem 1 (13 points)

(a) (5 points). A second order differential equation is given by

$$y'' + 3y' - 4y = 0.$$

Below is a list of function expressions containing two arbitrary constants c_1 and c_2 . Mark the expression which constitutes the general solution of the differential equation.

- $y(t) = c_1e^{-3t} + c_2e^t$
- $y(t) = c_1e^{-4t} + c_2e^t$
- $y(t) = c_1e^{-3t} + c_2e^{5t}$
- $y(t) = c_1e^t + c_2e^{5t}$
- $y(t) = c_1e^{2t} + c_2te^{2t}$
- $y(t) = c_1e^{-3t} \cos(3t) + c_2e^{-3t} \sin(3t)$
- $y(t) = c_1e^t \cos(6t) + c_2e^t \sin(6t)$
- $y(t) = c_1e^{-3t} \cos(2t) + c_2e^{-3t} \sin(2t)$

(b) (4 points). Mark the solution to the initial value problem

$$y'' + 3y' - 4y = 0, \quad y(0) = 7, \quad y'(0) = 2$$

among the following options:

- $y(t) = -\frac{5}{4}e^{-3t} + c_2e^t$
- $y(t) = 7e^{2t} - 3te^{2t}$
- $y(t) = e^{-4t} + 6e^t$
- $y(t) = 2e^{-3t} \cos(3t) + 3e^{-3t} \sin(3t)$
- $y(t) = 3e^{-3t} + c_2e^{5t}$
- $y(t) = 7e^t \cos(6t) + 2e^t \sin(6t)$
- $y(t) = -\frac{1}{4}e^t + \frac{5}{4}e^{5t}$
- $y(t) = e^{-3t} \cos(2t) + e^{-3t} \sin(2t)$

(c) (4 points). Consider the inhomogeneous differential equation

$$y'' + 3y' - 4y = t - 1.$$

Which one of the following functions is a particular solution to that equation?

- $2t^2 - t + 1$
- $\frac{3}{2}t + e^t$
- $t^2 - 2$
- $-\frac{1}{4}t + \frac{1}{16}$
- $t - 1$
- $5t - 3$
- $5t - \frac{1}{4}$
- $\frac{1}{5}t + \frac{1}{5}$

Problem 2 (7 points)

A plane curve is given by

$$\begin{aligned}x &= \sin(t) - e^t, \\y &= \cos(t) + e^t,\end{aligned}$$

where the parameter t runs through the real numbers.

(a) (3 points). The point $P = (-1, 2)$ lies on the curve. Which value of the parameter t corresponds to this point?

- | | | | |
|------------------------------------|---------------------------------|-----------------------------------|--|
| <input type="checkbox"/> -2 | <input type="checkbox"/> $-\pi$ | <input type="checkbox"/> 2π | <input type="checkbox"/> 2 |
| <input type="checkbox"/> $-\ln(2)$ | <input type="checkbox"/> 0 | <input type="checkbox"/> $\ln(2)$ | <input type="checkbox"/> $\frac{\pi}{2}$ |

(b) (4 points). What is the curvature of the curve for $t = 0$?

- | | | | |
|--------------------------------------|---|-----------------------------------|--|
| <input type="checkbox"/> -1 | <input type="checkbox"/> $-\frac{\pi}{2}$ | <input type="checkbox"/> π | <input type="checkbox"/> 1 |
| <input type="checkbox"/> $-\ln(2)/2$ | <input type="checkbox"/> 0 | <input type="checkbox"/> $\ln(1)$ | <input type="checkbox"/> $\frac{\pi}{4}$ |

Problem 3 (8 points)

A curve in space is given by

$$\begin{aligned}x &= 4 \sin(e^t), \\y &= 4 \cos(e^t), \\z &= 3e^t,\end{aligned}$$

where the parameter t runs through the positive real numbers.

(a) (3 points). What is the derivative y' ?

- | | | | |
|---|--|--|---|
| <input type="checkbox"/> $4e^t \cos(t)$ | <input type="checkbox"/> $-4e^t \cos(t)$ | <input type="checkbox"/> $-4e^t \sin(e^t)$ | <input type="checkbox"/> $4 \cos(e^t)$ |
| <input type="checkbox"/> $4e^t$ | <input type="checkbox"/> $-4e^t \sin(t)$ | <input type="checkbox"/> $-4 \cos(e^t)$ | <input type="checkbox"/> $4e^t \cos(e^t)$ |

(b) (5 points). What is the arc length of the curve from $t = 0$ to $t = \ln(3)$?

- | | | | |
|-------------------------------|-------------------------------------|---------------------------------|---------------------------------------|
| <input type="checkbox"/> 2 | <input type="checkbox"/> $5(e - 1)$ | <input type="checkbox"/> 2π | <input type="checkbox"/> $27\sqrt{3}$ |
| <input type="checkbox"/> 10 | <input type="checkbox"/> $\ln(3)$ | <input type="checkbox"/> e^3 | <input type="checkbox"/> -2 |

Problem 4 (7 points) A function is defined by

$$f(x) = \frac{1}{x^2 + x + 1}.$$

(a) (3 points). What is the derivative $f'(x)$?

$\frac{1}{2x+1}$

$\frac{-2x-1}{(x^2+x+1)^2}$

$\frac{2x+1}{(x^2+x+1)^{-\frac{1}{2}}}$

$(x^2 + x + 1)^{-\frac{1}{2}}$

$-\frac{1}{(x^2+x+1)^2}$

$\ln(x^2 + x + 1)$

(b) (4 points). Which one of the polynomials below is the second order Taylor polynomial for f at the point $x = 0$?

$1 + x - \frac{3}{2}x^2$

$1 - x + x^3$

$1 + 4x - 5x^2$

$\frac{1}{2} + \frac{1}{2}x - \frac{1}{6}x^2$

$1 - x$

$1 - x^2$

Problem 5 (7 points)

Consider the differential equation

$$\frac{dy}{dx} = -3\frac{e^{-x}}{y}, \quad y > 0.$$

There is a unique solution y with initial value $y(0) = 2$. Answer the following questions regarding the solution:

(a) (3 points). What is the derivative $y'(0)$?

-2

$\frac{2}{3}$

2

$-\frac{3}{2}$

1

3

(b) (4 points). What is the function value $y(\ln(2))$?

5

$\frac{1}{3}$

$\ln(2)$

$e + 1$

1

$\sqrt{2e + 2}$

Problem 6 (6 points)

Two complex numbers are given by

$$z_1 = \frac{4 - 3i}{2 + i} + 1 + i, \quad z_2 = (e^{3 + \frac{\pi}{3}i})^3$$

(a) (3 points). What is z_1 written in standard form?

- $2 + 3i$ $\frac{1}{4} + \frac{5}{4}i$ $-3i$ $2 - 3i$
 $1 + i$ $\frac{3}{2} + \frac{2}{3}i$ $2 - i$

(b) (3 points). What is z_2 written in standard form?

- $1 + e^3i$ $1 - e^2i$ $-e^9$ $e + e^2i$
 e^6 $1 + e^2i$ e^6i $1 + 3i$

Problem 7 (14 points)

A function is given by

$$f(x, y) = (x + y^2) \ln(x + y^2) - x$$

(a) (2 point) The function f has a domain of definition. Mark the corresponding box, if it consists of all points (x, y) for which

- $x \geq -y^2$ $x > y^2$ $y^2 > -x$ $x \geq 0$

(b) (3 point) Which of the following expressions is the second order partial derivative $f_{xy}(x, y)$?

- $\frac{1}{x+y^2}$ $1 - \frac{1}{(x+y^2)^2}$ $2 \ln(x + y^2)$ $\frac{2y}{x+y^2}$

(c) (3 point) Which of the following vectors is equal to the functions gradient ∇f in the point $P = (0, 1)$?

- $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(d) (3 point) Which of the following numbers is equal to the directional derivative $D_u f(P)$ of the function f in the point $P = (0, 1)$ and in the direction determined by the unit vector

$$u = \frac{3}{5}i + \frac{4}{5}j = \left(\frac{3}{5}, \frac{4}{5} \right) ?$$

- $\frac{4}{5}$ 1 $\frac{6}{5}$ $\frac{7}{5}$ $\frac{8}{5}$

(e) (3 point) Which of the following equations describes the tangent plane to the graph of the function f in the point $Q = (0, 1, 0)$?

$z = x - 2y - 2$
 $z = 2y - 2$

$z = 2y$
 $z = x - 2$

Problem 8 (10 points)

A surface \mathcal{F} in space is determined by the equation

$$F(x, y, z) = x^2 - 2y^2 - z^2 = 1.$$

(a) (2 point) Which of the following points are contained in the surface \mathcal{F} ?

$(2, -1, 1)$
 $(1, 1, 2)$

$(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$
 $(1, 0)$

$(1, 2, 1)$
 $(0, -\frac{1}{2}, 0)$

(b) (2 point) Which of the following vectors is normal to/orthogonal to the tangent plane at the point $P = (2, 1, -1)$ of the surface \mathcal{F} ?

$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$

$\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

(c) (2 point) Which of the following points is contained in the tangent plane at the point $P = (2, 1, -1)$ of the surface \mathcal{F} ?

$(0, 0, 0)$

$(3, 2, 1)$

$(-3, 1, 9)$

$(4, 4, -1)$

(d) (4 point) In which of the following points Q on the surface \mathcal{F} is the tangent plane of the surface \mathcal{F} at Q parallel to the plane $z = 2x - 2y$?

$Q = (2, 1, -1)$
 $Q = (3, 2, 0)$

$Q = (1, 0, 0)$
 $Q = (\sqrt{2}, \frac{1}{\sqrt{2}}, 0)$

$Q = (-\sqrt{2}, \frac{1}{\sqrt{2}}, 0)$
 $Q = (2, 1, 1)$

Problem 9 (8 points)

A domain \mathcal{R} in the plane consists of all points (x, y) , that satisfy the inequalities

$$0 \leq x, \quad 4 \leq x^2 + y^2 \leq 5.$$

Mark the correct value of the plane integral

$$\iint_{\mathcal{R}} x \sqrt{x^2 + y^2} dA.$$

- $\frac{7}{4}$ $\frac{\sqrt{2}}{3}$ $\frac{16}{5}$ $\frac{5\pi}{3}$ $\frac{9}{2}$ 2

Problem 10 (7 points)

A thin plate is covering precisely a domain \mathcal{R} in the plane. The domain \mathcal{R} consists of all points (x, y) , that satisfy the inequalities

$$0 \leq x, \quad 0 \leq y \leq 1 - x^2.$$

The mass density of the plate is $\delta(x, y) = 3$. What is the total mass of the plate?

- $\frac{3}{4}$ 1 $\frac{3}{2}$ 2 $\frac{9}{4}$ $\frac{5}{2}$

Problem 11 (8 points)

A body \mathcal{T} in space has the form of a tetrahedron bounded by four planes; it is given by

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad x + y + z \leq 1,$$

and it has the mass density

$$\delta(x, y, z) = 1 - x.$$

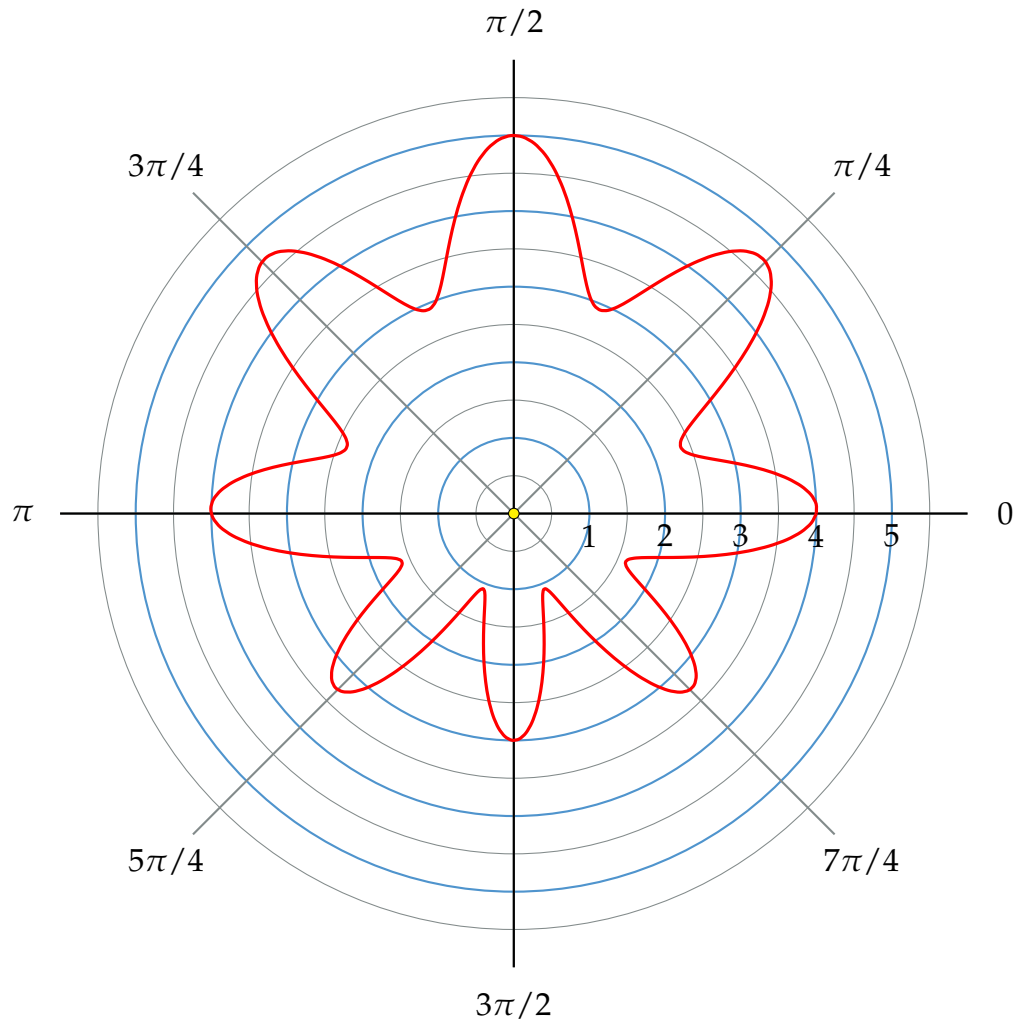
Which of the following values is equal to the total mass of the body \mathcal{T} ?

- $\frac{1}{4}$ $\frac{1}{8}$ $\frac{1}{12}$ $\frac{1}{16}$ $\frac{1}{24}$ $\frac{1}{32}$

Problem 12 (5 points)

The figure below shows the polar graph of the polar function

$$r = f(\theta), \quad 0 \leq \theta \leq 2\pi.$$



One of the expressions for f in the following list corresponds to the above figure. Which one?

$f(\theta) = 3 + \cos^2(16\theta) + \sin(\theta)$

$f(\theta) = 4 + \cos(8\theta) \sin(\theta)$

$f(\theta) = 4 \cos(16\theta) \sin(\theta)$

$f(\theta) = 4 \cos(8\theta) + \sin(\theta)$

$f(\theta) = 3 + \cos(8\theta) + \sin(\theta)$

$f(\theta) = 4 \cos(8\theta) \sin(\theta/2)$