Reexam in Calculus

First Year at the Faculty of Engineering and Science and the Technical Faculty of IT and Design

26. Februar 2020

The present exam set consists of 8 numbered pages with 12 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use books, notes, etc. It is **not** allowed to use electronic devices.

Problem 1 (13 points)

(a) (5 points). A second order differential equation is given by

$$y'' + 3y' - 4y = 0.$$

Below is a list of function expressions containing two arbitrary constants c_1 and c_2. Mark the expression which constitutes the general solution of the differential equation.

- ☐ $y(t) = c_1 e^{-3t} + c_2 e^{t}$
- ☑ $y(t) = c_1 e^{-4t} + c_2 e^{t}$
- ☐ $y(t) = c_1 e^{-3t} + c_2 e^{5t}$
- ☐ $y(t) = c_1 e^{t} + c_2 e^{5t}$
- ☐ $y(t) = c_1 e^{2t} + c_2 t e^{2t}$
- ☐ $y(t) = c_1 e^{-3t}\cos(3t) + c_2 e^{-3t}\sin(3t)$
- ☐ $y(t) = c_1 e^{t}\cos(6t) + c_2 e^{t}\sin(6t)$
- ☐ $y(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$

(b) (4 points). Mark the solution to the initial value problem

$$y'' + 3y' - 4y = 0, \quad y(0) = 7, \quad y'(0) = 2$$

among the following options:

☐ $y(t) = -\frac{5}{4}e^{-3t} + c_2 e^{t}$	☐ $y(t) = 7e^{2t} - 3te^{2t}$
☑ $y(t) = e^{-4t} + 6e^{t}$	☐ $y(t) = 2e^{-3t}\cos(3t) + 3e^{-3t}\sin(3t)$
☐ $y(t) = 3e^{-3t} + c_2 e^{5t}$	☐ $y(t) = 7e^{t}\cos(6t) + 2e^{t}\sin(6t)$
☐ $y(t) = -\frac{1}{4}e^{t} + \frac{5}{4}e^{5t}$	☐ $y(t) = e^{-3t}\cos(2t) + e^{-3t}\sin(2t)$

(c) (4 points). Consider the inhomogeneous differential equation

$$y'' + 3y' - 4y = t - 1.$$

Which one of the following functions is a particular solution to that equation?

☐ $2t^2 - t + 1$	☐ $\frac{3}{2}t + e^{t}$	☐ $t^2 - 2$	☑ $-\frac{1}{4}t + \frac{1}{16}$
☐ $t - 1$	☐ $5t - 3$	☐ $5t - \frac{1}{4}$	☐ $\frac{1}{5}t + \frac{1}{5}$

Problem 2 (7 points)

A plane curve is given by

$$x = \sin(t) - e^t,$$
$$y = \cos(t) + e^t,$$

where the parameter t runs through the real numbers.

(a) (3 points). The point $P = (-1, 2)$ lies on the curve. Which value of the parameter t corresponds to this point?

- ☐ -2
- ☐ $-\pi$
- ☐ 2π
- ☐ 2
- ☐ $-\ln(2)$
- ☑ 0
- ☐ $\ln(2)$
- ☐ $\frac{\pi}{2}$

(b) (4 points). What is the curvature of the curve for $t = 0$?

- ☐ -1
- ☐ $-\frac{\pi}{2}$
- ☐ π
- ☑ 1
- ☐ $-\ln(2)/2$
- ☐ 0
- ☐ $\ln(1)$
- ☐ $\frac{\pi}{4}$

Problem 3 (8 points)

A curve in space is given by

$$x = 4\sin(e^t),$$
$$y = 4\cos(e^t),$$
$$z = 3e^t,$$

where the parameter t runs through the positive real numbers.

(a) (3 points). What is the derivative y'?

- ☐ $4e^t\cos(t)$
- ☐ $-4e^t\cos(t)$
- ☑ $-4e^t\sin(e^t)$
- ☐ $4\cos(e^t)$
- ☐ $4e^t$
- ☐ $-4e^t\sin(t)$
- ☐ $-4\cos(e^t)$
- ☐ $4e^t\cos(e^t)$

(b) (5 points). What is the arc length of the curve from $t = 0$ to $t = \ln(3)$?

- ☐ 2
- ☐ $5(e-1)$
- ☐ 2π
- ☐ $27\sqrt{3}$
- ☑ 10
- ☐ $\ln(3)$
- ☐ e^3
- ☐ -2

Problem 4 (7 points) A function is defined by

$$f(x) = \frac{1}{x^2+x+1}.$$

(a) (3 points). What is the derivative $f'(x)$?

- ☐ $\frac{1}{2x+1}$
- ☑ $\frac{-2x-1}{(x^2+x+1)^2}$
- ☐ $\frac{2x+1}{(x^2+x+1)^{-\frac{1}{2}}}$
- ☐ $(x^2+x+1)^{-\frac{1}{2}}$
- ☐ $-\frac{1}{(x^2+x+1)^2}$
- ☐ $\ln(x^2+x+1)$

(b) (4 points). Which one of the polynomials below is the second order Taylor polynomial for f at the point $x = 0$?

- ☐ $1 + x - \frac{3}{2}x^2$
- ☐ $1 - x + x^3$
- ☐ $1 + 4x - 5x^2$
- ☐ $\frac{1}{2} + \frac{1}{2}x - \frac{1}{6}x^2$
- ☑ $1 - x$
- ☐ $1 - x^2$

Problem 5 (7 points)

Consider the differential equation

$$\frac{dy}{dx} = -3\frac{e^{-x}}{y}, \quad y > 0.$$

There is a unique solution y with initial value $y(0) = 2$. Answer the following questions regarding the solution:

(a) (3 points). What is the derivative $y'(0)$?

- ☐ -2
- ☐ $\frac{2}{3}$
- ☐ 2
- ☑ $-\frac{3}{2}$
- ☐ 1
- ☐ 3

(b) (4 points). What is the function value $y(\ln(2))$?

- ☐ 5
- ☐ $\frac{1}{3}$
- ☐ $\ln(2)$
- ☐ $e + 1$
- ☑ 1
- ☐ $\sqrt{2e+2}$

Problem 6 (6 points)

Two complex numbers are given by

$$z_1 = \frac{4-3i}{2+i} + 1 + i, \quad z_2 = (e^{3+\frac{\pi}{3}i})^3$$

(a) (3 points). What is z_1 written in standard form?

- ☐ $2+3i$
- ☐ $\frac{1}{4}+\frac{5}{4}i$
- ☐ $-3i$
- ☐ $2-3i$
- ☐ $1+i$
- ☐ $\frac{3}{2}+\frac{2}{3}i$
- ☑ $2-i$

(b) (3 points). What is z_2 written in standard form?

- ☐ $1+e^3 i$
- ☐ $1-e^2 i$
- ☑ $-e^9$
- ☐ $e+e^2 i$
- ☐ e^6
- ☐ $1+e^2 i$
- ☐ $e^6 i$
- ☐ $1+3i$

Problem 7 (14 points)

A function is given by

$$f(x,y) = (x+y^2)\ln(x+y^2) - x$$

(a) (2 point) The function f has a domain of definition. Mark the correspon-ding box, if it consists of all points (x,y) for which

- ☐ $x \geq -y^2$
- ☐ $x > y^2$
- ☑ $y^2 > -x$
- ☐ $x \geq 0$

(b) (3 point) Which of the following expressions is the second order partial derivative $f_{xy}(x,y)$?

- ☐ $\frac{1}{x+y^2}$
- ☐ $1-\frac{1}{(x+y^2)^2}$
- ☐ $2\ln(x+y^2)$
- ☑ $\frac{2y}{x+y^2}$

(c) (3 point) Which of the following vectors is equal to the functions gradient ∇f in the point $P=(0,1)$?

- ☐ $\begin{bmatrix}1\\2\end{bmatrix}$
- ☑ $\begin{bmatrix}0\\2\end{bmatrix}$
- ☐ $\begin{bmatrix}1\\1\end{bmatrix}$
- ☐ $\begin{bmatrix}2\\0\end{bmatrix}$

(d) (3 point) Which of the following numbers is equal to the directional derivative $D_u f(P)$ of the function f in the point $P=(0,1)$ and in the direction determined by the unit vector

$$u = \frac{3}{5}i + \frac{4}{5}j = \left(\frac{3}{5}, \frac{4}{5}\right) ?$$

- ☐ $\frac{4}{5}$
- ☐ 1
- ☐ $\frac{6}{5}$
- ☐ $\frac{7}{5}$
- ☑ $\frac{8}{5}$

(e) (3 point) Which of the following equations describes the tangent plane to the graph of the function f in the point $Q = (0, 1, 0)$?

- ☐ $z = x - 2y - 2$
- ☑ $z = 2y - 2$
- ☐ $z = 2y$
- ☐ $z = x - 2$

Problem 8 (10 points)

A surface $\mathcal{F}$ in space is determined by the equation

$$F(x, y, z) = x^2 - 2y^2 - z^2 = 1.$$

(a) (2 point) Which of the following points are contained in the surface $\mathcal{F}$?

- ☑ $(2, -1, 1)$
- ☐ $(1, 1, 2)$
- ☐ $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$
- ☐ $(1, 0)$
- ☐ $(1, 2, 1)$
- ☐ $(0, -\frac{1}{2}, 0)$

(b) (2 point) Which of the following vectors is normal to/orthogonal to the tangent plane at the point $P = (2, 1, -1)$ of the surface $\mathcal{F}$?

- ☐ $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$
- ☑ $\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$
- ☐ $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

(c) (2 point) Which of the following points is contained in the tangent plane at the point $P = (2, 1, -1)$ of the surface $\mathcal{F}$?

- ☐ $(0, 0, 0)$
- ☐ $(3, 2, 1)$
- ☑ $(-3, 1, 9)$
- ☐ $(4, 4, -1)$

(d) (4 point) In which of the following points Q on the surface $\mathcal{F}$ is the tangent plane of the surface $\mathcal{F}$ at Q parallel to the plane $z = 2x - 2y$?

- ☐ $Q = (2, 1, -1)$
- ☐ $Q = (3, 2, 0)$
- ☐ $Q = (1, 0, 0)$
- ☐ $Q = (\sqrt{2}, \frac{1}{\sqrt{2}}, 0)$
- ☐ $Q = (-\sqrt{2}, \frac{1}{\sqrt{2}}, 0)$
- ☑ $Q = (2, 1, 1)$

Problem 9 (8 points)

A domain $\mathcal{R}$ in the plane consists of all points (x, y), that satisfy the inequalities

$$0 \leq x, \quad 4 \leq x^2 + y^2 \leq 5.$$

Mark the correct value of the plane integral

$$\iint_{\mathcal{R}} x\sqrt{x^2 + y^2}\, dA.$$

☐ $\frac{7}{4}$ ☐ $\frac{\sqrt{2}}{3}$ ☐ $\frac{16}{5}$ ☐ $\frac{5\pi}{3}$ ☑ $\frac{9}{2}$ ☐ 2

Problem 10 (7 points)

A thin plate is covering precisely a domain $\mathcal{R}$ in the plane. The domain $\mathcal{R}$ consists of all points (x, y), that satisfy the inequalities

$$0 \leq x, \quad 0 \leq y \leq 1 - x^2.$$

The mass density of the plate is $\delta(x, y) = 3$. What is the total mass of the plate?

☐ $\frac{3}{4}$ ☐ 1 ☐ $\frac{3}{2}$ ☑ 2 ☐ $\frac{9}{4}$ ☐ $\frac{5}{2}$

Problem 11 (8 points)

A body $\mathcal{T}$ in space has the form of a tetrahedron bounded by four planes; it is given by

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad x + y + z \leq 1,$$

and it has the mass density

$$\delta(x, y, z) = 1 - x.$$

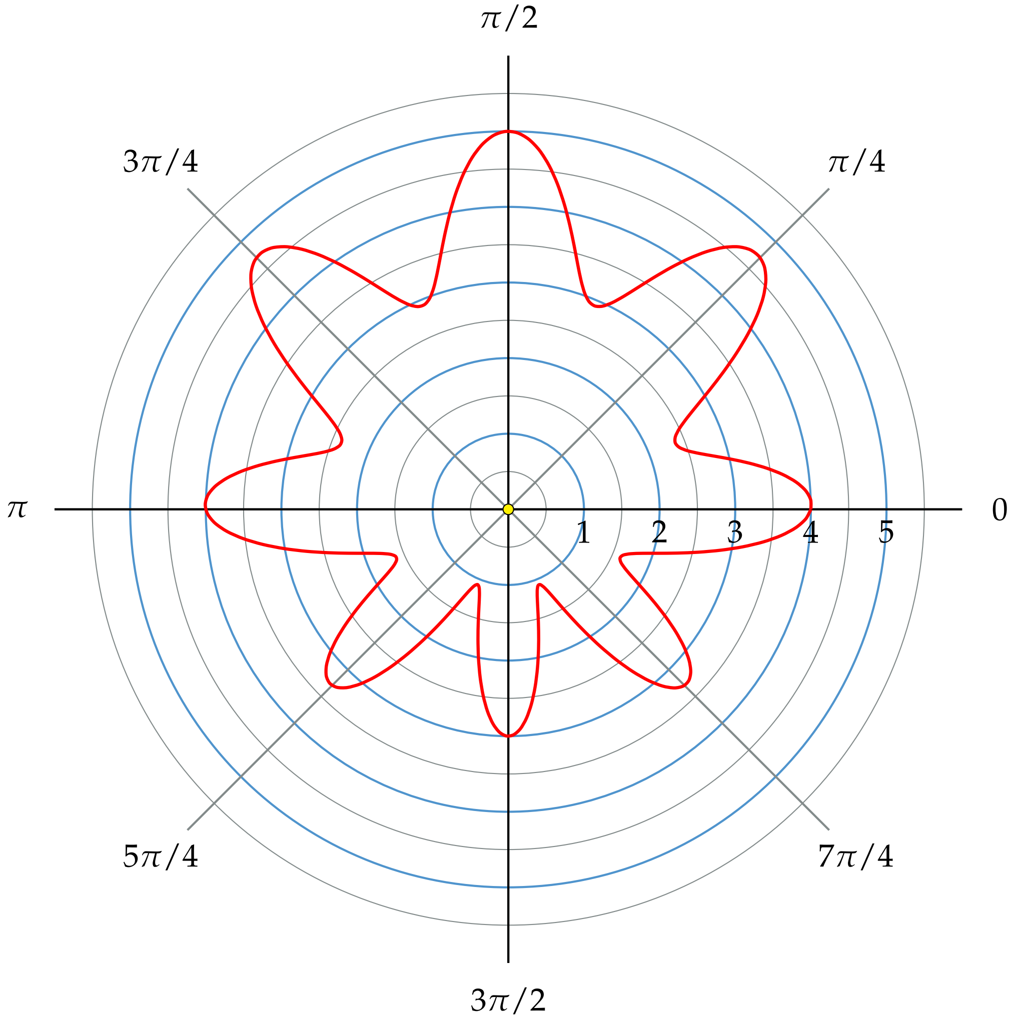
Which of the following values is equal to the total mass of the body $\mathcal{T}$?

☐ $\frac{1}{4}$ ☑ $\frac{1}{8}$ ☐ $\frac{1}{12}$ ☐ $\frac{1}{16}$ ☐ $\frac{1}{24}$ ☐ $\frac{1}{32}$

Problem 12 (5 points)

The figure below shows the polar graph of the polar function

$$r = f(\theta), \quad 0 \leq \theta \leq 2\pi.$$



One of the expressions for f in the following list corresponds to the above figure. Which one?

- ☐ $f(\theta) = 3 + \cos^2(16\theta) + \sin(\theta)$
- ☐ $f(\theta) = 4 + \cos(8\theta)\sin(\theta)$
- ☐ $f(\theta) = 4\cos(16\theta)\sin(\theta)$
- ☐ $f(\theta) = 4\cos(8\theta) + \sin(\theta)$
- ☑ $f(\theta) = 3 + \cos(8\theta) + \sin(\theta)$
- ☐ $f(\theta) = 4\cos(8\theta)\sin(\theta/2)$