For at finde den danske version af prøven, begynd i den modsatte ende!

Please disregard the Danish version on the back if you participate in this English version of the exam.

Exam in Calculus

First Year at the Technical Faculty for IT og Design, the Faculty of Medicine and the Faculty of Engineering and Science

August 24, 2018

This test consists of 8 numbered pages and 12 multiple choice problems. A number of points are assigned to each problem. The entire test consists of 100 points in total.

It is allowed to use books, notes, etc. It is **not allowed** to use any **electronic devices**.

Your answers must be marked on these sheets. In each subproblem you should **only mark one of the listed choices**. The evaluation is solely based on your marked answers on these sheets.

Remember to write your **full name** and **student number** below. Moreover, please mark the team that you participate in.

Good luck!

NAME: ________________________________

STUDENT NUMBER: ________________________________

- ☐ Team 1: LAND – ST — Horia Cornean
- ☐ Team LAN (Copenhagen) — Iver Ottosen
- ☐ Team BBIO – MOE (Copenhagen) — Oliver Matte

with solutions

Problem 1 (6 points)

A function is given by

$$f(x,y) = \frac{x - 3y^2 - 1}{y^2 - x}$$

for real variables x and y.

(a) (2 points) The domain of f consists of all points (x,y) that satisfy

☐ $y \neq \sqrt{\lvert x \rvert}$	☐ $x > 0,\ y < 0$	☐ $3y^2 = x + 1$
☑ $y^2 \neq x$	☐ $x < 0,\ y > 0$	☐ $y = \sqrt{x}$

(b) (4 points) Mark the correct expression for the level curve $f(x,y) = -3$.

- ☐ A parabola $x = 2y^2 + 1$
- ☐ A parabola $x = 3y^2 - 1$
- ☐ A circle with center $(-1, 0)$ and radius 1
- ☐ A circle with center $(-1, 0)$ and radius 1, excluding the point $(0, 0)$
- ☑ A straight line $x = -\frac{1}{2}$
- ☐ A straight line $x = -\frac{1}{2}$, excluding the points $(-\frac{1}{2}, -\frac{1}{\sqrt{2}})$ and $(-\frac{1}{2}, \frac{1}{\sqrt{2}})$

Problem 2 (6 points)

A parametrized space curve is given by

$$\mathbf{r}(t) = \left\langle e^t,\ e^{5t},\ e^{(t^2)} \right\rangle$$

where the parameter t can take any real value.

(a) (3 points) What is the curve's speed at $t = 0$?

☐ 0	☐ $\sqrt{15}$	☐ 5
☐ $2\sqrt{13}$	☑ $\sqrt{26}$	☐ $\sqrt{30}$

(b) (3 points) Which of the following vectors agrees with the curve's acceleration vector at $t = 2$?

☑ $\langle e^2,\ 25e^{10},\ 18e^4 \rangle$	☐ $\langle e^2,\ e^{10},\ e^4 \rangle$	☐ $\langle 0,\ 0,\ 0 \rangle$
☐ $\langle 1,\ 25,\ 10 \rangle$	☐ $\langle e^2,\ 5e^{10},\ 4e^4 \rangle$	☐ $\langle e^2,\ 25e^{10},\ 16e^4 \rangle$

Problem 3 (6 points)

Three complex numbers are given by

$$z_1 = 1 + 2i, \quad z_2 = 2e^{\frac{3\pi}{2}i} \quad \text{and} \quad z_3 = 2 + 2i.$$

(a) (3 points) What is $z_1 z_3$ on standard form?

- ☐ 0
- ☐ $2 + 4i$
- ☐ $1 + i$
- ☐ $6 + 4i$
- ☐ $2\sqrt{10}e^{\frac{2\pi}{3}i}$
- ☑ $-2 + 6i$

(b) (3 points) What is $\frac{z_2}{z_3}$ on polar form?

- ☐ $\frac{1}{2}e^{\frac{3\pi}{2}i}$
- ☐ 2
- ☑ $\frac{1}{\sqrt{2}}e^{\frac{5\pi}{4}i}$
- ☐ $4e^{\frac{5\pi}{4}i}$
- ☐ $2e^{\frac{7\pi}{4}i}$
- ☐ $e^{\frac{3\pi}{2}i}$

Problem 4 (10 points)

(a) (5 points) A homogeneous second order differential equation is given by

$$y'' + 2y = 0.$$

Several functions are given below, where c_1 and c_2 are arbitrary real constants. Mark the function which agrees with the general solution of the differential equation.

- ☐ $y(t) = c_1 e^{-\sqrt{2}t} + c_2 e^{\sqrt{2}t}$
- ☐ $y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$
- ☑ $y(t) = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t)$
- ☐ $y(t) = c_1 \sin(2t) + c_2 \cos(2t)$
- ☐ $y(t) = c_1 e^{-2t} + c_2 e^{2t}$
- ☐ $y(t) = c_1 t^2 + c_2 \sqrt{2}t$
- ☐ $y(t) = c_1 e^{-t} + c_2 t e^{-t}$
- ☐ $y(t) = c_1 e^{-4t} + c_2 e^{2t}$

(b) (5 points) Mark a particular solution x_p of the inhomogeneous differential equation

$$x'' + 2x = 4t^2,$$

from the following list of functions.

- ☐ $x_p(t) = 2t^2$
- ☐ $x_p(t) = -4t^2 - t - 2$
- ☐ $x_p(t) = -4t^2$
- ☑ $x_p(t) = 2t^2 - 2$
- ☐ $x_p(t) = t^2 + t + 1$
- ☐ $x_p(t) = 3t^4 + 2t^2 + 2t - 1$
- ☐ $x_p(t) = 2t^2 - t - 2$
- ☐ $x_p(t) = \cos(t) + \sin(t) + t^2 - 2$

Problem 5 (8 points)

Mark whether the following statements regarding curvature are true or false.

(a) (2 points) A straight line can have positive curvature.

☐ True ☑ False

(b) (2 points) A cirle with radius R has constant curvature $\frac{1}{R}$.

☑ True ☐ False

In subproblem (c) and (d) consider the following: Two particles $\mathbf{r}_A(t)$ and $\mathbf{r}_B(t)$ move along the same curve which contains a point P.

(c) (2 points) *The speed* of $\mathbf{r}_A(t)$ is twice as large as the speed of $\mathbf{r}_B(t)$. Then the curvature of $\mathbf{r}_A(t)$ at the point P is identical to the curvature of $\mathbf{r}_B(t)$ at P.

☑ True ☐ False

(d) (2 points) *The acceleration* of $\mathbf{r}_A(t)$ is twice as large as the acceleration of $\mathbf{r}_B(t)$. Then the curvature of $\mathbf{r}_A(t)$ at the point P is twice as large as the curvature of $\mathbf{r}_B(t)$ at P.

☐ True ☑ False

Problem 6 (6 points)

A region $\mathcal{R}$ in the plane consists of all points within and on the triangle with corners at the points $A = (0, 0)$, $B = (0, 1)$ and $C = (3, 0)$. A body with density $\delta(x, y) = 2xy^2$ covers the region $\mathcal{R}$.

(a) (3 points) Which of the following inequalities determine that a point with coordinates (x, y) belongs to $\mathcal{R}$?

☐ $0 \leq x \leq 3, \quad 0 \leq y \leq 1$ ☑ $0 \leq x \leq 3, \quad 0 \leq y \leq 1 - \frac{1}{3}x$

☐ $0 \leq x \leq 9 - 3y, \quad 0 \leq y \leq 3$ ☐ $y = 3 - 3x$

☐ $0 \leq x \leq 1, \quad 0 \leq y \leq 3 - 3x$ ☐ $x = 3, \quad y = 1$

(b) (3 points) What is the correct formula that determines the mass of the body?

☐ $\int_0^3 \int_0^{9-3y} 2xy^2 \, dxdy$ ☑ $\int_0^3 \int_0^{1-\frac{1}{3}x} 2xy^2 \, dydx$ ☐ $\int_0^1 \int_0^{3-3x} 1 \, dydx$

☐ $\int_0^1 \int_0^{3-3x} 2xy^2 \, dydx$ ☐ $\int_0^3 \int_0^{1-\frac{1}{3}x} 1 \, dydx$ ☐ $\int_0^1 \int_0^3 2xy^2 \, dxdy$

Problem 7 (6 points)

A region $\mathcal{R}$ in the plane consists of all points with coordinates (x, y) that satisfy the following two inequalities:

$$1 \leq \sqrt{x^2 + y^2} \leq 2.$$

Mark the correct value of the double integral

$$\int_{\mathcal{R}} 4(x^2 + y^2)\, dA.$$

- ☐ 0
- ☐ 2π
- ☑ 30π
- ☐ -2π
- ☐ $15\pi/2$
- ☐ 47

Problem 8 (10 points)

A surface $\mathcal{F}$ is defined by the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = -2\sin(z) + yz - x^2 y + y^2 - 1.$$

(a) (5 points) Which of the listed equations determines the tangent plane of $\mathcal{F}$ at the point $P = (0, 1, 0)$?

- ☐ $0 = x + \frac{3}{2}y + z$
- ☐ $z = -\frac{1}{2}y - 1$
- ☐ $z = 1$
- ☑ $z = 2y - 2$
- ☐ $2 = -2y + 2z$
- ☐ $0 = -x - y + 3z$

(b) (5 points) From the equation $F(x, y, z) = 0$, what is the partial derivative $\partial z / \partial y$ evaluated at the point P?

- ☐ -3
- ☐ -1
- ☐ 0
- ☑ 2
- ☐ π
- ☐ 3

Problem 9 (17 points)

A function is given by

$$f(x, y) = \arctan(2x + y) = \tan^{-1}(2x + y),$$

where the variables x and y can take any real value.

(a) (4 points) Mark whether the following statement is true or false: The function f has at least one critical point.

☐ True ☑ False

(b) (4 points) Mark whether the following statement is true or false: The function f has a global maximum.

☐ True ☑ False

(c) (4 points) What is the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (-1, 1)$ and in the direction given by the unit vector $\mathbf{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$?

☐ -2 ☐ $\frac{3}{\sqrt{2}}$ ☑ $\frac{3\sqrt{2}}{4}$

☐ $\frac{1}{\sqrt{2}}$ ☐ $5\sqrt{2}$ ☐ 4

(d) (5 points) Which of the following unit vectors points in the direction of steepest ascend for f at the point P (the direction $\mathbf{v}$ for which $D_{\mathbf{v}}f(P)$ is as large as possible)?

☐ $\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$ ☐ $\langle \frac{2\sqrt{5}}{5}, -\frac{1}{\sqrt{5}} \rangle$ ☐ $\langle 0, -1 \rangle$

☐ $\langle 0, 1 \rangle$ ☐ $\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \rangle$ ☐ $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

☐ $\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle$ ☐ $\langle 1, 0 \rangle$ ☑ $\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$

Problem 10 (9 points)

A function is given by

$$f(x) = \ln(\sqrt{2}x + 1)$$

for $x > -2^{-1/2}$.

(a) (5 points) Mark the expression which agrees with $f''(x)$ (*hint: remember to use the chain rule*).

☐ $2\ln(\sqrt{2}x+1)$ ☑ $\frac{-2}{(\sqrt{2}x+1)^2}$ ☐ $\frac{1}{\sqrt{2}}$

☐ $\frac{\sqrt{2}}{\ln(\sqrt{2}x+1)}$ ☐ $\frac{2}{2x^2+2\sqrt{2}x+1}$ ☐ $\frac{-2\ln(\sqrt{2}x+1)}{(\sqrt{2}x+1)^2}$

(b) (4 points) Which of the following polynomials agrees with the second order Taylor polynomial of f about $x = 0$?

☐ $1 + \sqrt{2}x^2$ ☐ $1 - x^2$ ☐ $\sqrt{2}x + \frac{1}{\sqrt{2}}x^2$

☐ $1 + \sqrt{2}x + \sqrt{2}x^2$ ☐ $1 + 2x - x^2$ ☐ $\sqrt{2}x + 2x^2$

☐ $-2x + \sqrt{2}x^2$ ☑ $\sqrt{2}x - x^2$ ☐ $2 - \sqrt{2}x - \sqrt{2}x^2$

Problem 11 (11 points)

A planar curve is given by

$$x = t + 3t^2,$$
$$y = 3t - t^2.$$

(a) (2 points) For which value of the parameter t does the curve pass through the point $P = (4, 2)$?

☐ $-\pi$ ☐ -1 ☐ $-\frac{\pi}{4}$ ☐ 0 ☑ 1

(b) (4 points) What is the curvature of the curve at the point P?

☑ $\frac{\sqrt{2}}{25}$ ☐ $\frac{3}{50\sqrt{50}}$ ☐ 0 ☐ $\frac{\sqrt{2}}{100}$ ☐ $\frac{20}{\sqrt{5}}$

(c) (5 points) For which value of the parameter t is the curvature maximal?

☐ -3 ☑ 0 ☐ 2π

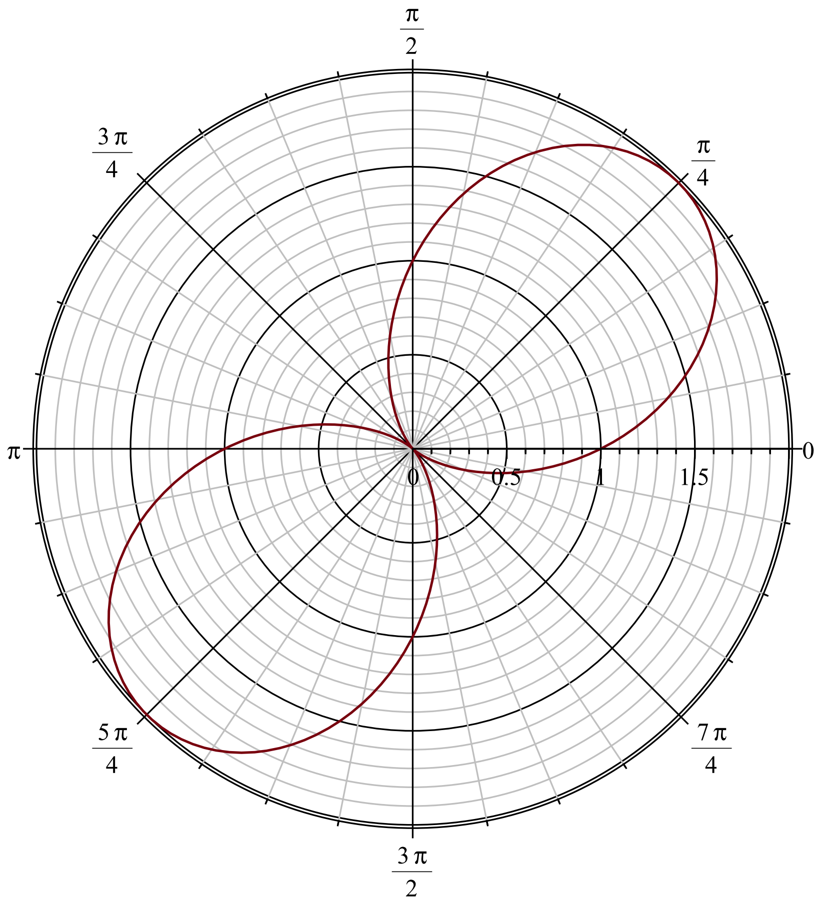
☐ -1 ☐ 1 ☐ 7

Problem 12 (5 points)

The figure below shows the graph of a function

$$r = f(\theta), \quad 0 \leq \theta \leq 2\pi,$$

in polar coordinates.



Which one of the functions below gives rise to that graph?

- [x] $f(\theta) = \sin(2\theta) + 1$
- [ ] $f(\theta) = \theta^2 + 1$
- [ ] $f(\theta) = \cos(4\theta) - 1$
- [ ] $f(\theta) = \cos(2\theta)\sin(\theta)$
- [ ] $f(\theta) = \sin(\theta) - \cos(\theta)$
- [ ] $f(\theta) = 2 - \sin(2\theta)$