Reexam in Calculus

20. february 2019

Exercise 1 (6 point)

A function is defined by

$$f(x,y) = 1 + \frac{y^2}{x^2}$$

for real variables x and y.

(a) (3 point) The domain of definition of f consists of all points (x,y) which obey

☐ $x < 0$ ☑ $x \neq 0$ ☐ $yx \neq 0$

☐ $y \neq 0$ ☐ $x \neq 0$ og $y > 0$ ☐ $y^2 = x^2$

(b) (3 point) What is the level set $f(x,y) = 2$?

☐ A parabola $x = y^2 + 1$

☐ A parabola $x = y^2 - 1$

☐ A circle with center at origin and radius 1

☐ A circle with center at origin and radius 2

☑ Two straight lines $x = \pm y$ without the origin.

Exercise 2 (6 point)

A parametric curve in space is given by

$$\mathbf{r}(t) = \left\langle t,\ \ -t^2,\ \ e^t \right\rangle$$

where the parameter t can be any real number.

(a) (3 point) What is the velocity?

☐ $\langle 1, -2t, e^t \rangle$ ☐ $\sqrt{5+e}$ ☑ $\sqrt{1+4t^2+e^{2t}}$

☐ $\sqrt{1-2t+e^t}$ ☐ $\sqrt{4+t^2}$ ☐ $\sqrt{1-4t^2+e^{2t}}$

(b) (3 point) Which of the following vectors represent the acceleration vector at $t = 0$?

☐ $\langle 0,\ -1,\ 0 \rangle$ ☐ $\langle 0,\ -2,\ 0 \rangle$ ☑ $\langle 0,\ -2,\ 1 \rangle$

☐ $\langle 0,\ -4,\ 0 \rangle$ ☐ $\langle -4,\ 0,\ 1 \rangle$ ☐ $\langle 0, -2, e \rangle$

Exercise 3 (6 point)

Three complex numbers are given by

$$z_1 = 1 - i, \quad z_2 = 2i^2 \quad \text{og} \quad z_3 = 1 + i.$$

(a) (3 point) What is $z_1 + z_2$ in polar form?

☐ 0	☐ $-2e^{i\pi/4}$	☐ $\sqrt{2}e^{-i\pi/4}$
☐ $2e^{i\pi/4}$	☑ $\sqrt{2}e^{\frac{5\pi}{4}i}$	☐ $\sqrt{2}e^{i\pi/2}$

(b) (3 point) What is $\frac{z_1}{z_3}$ in standard form?

☐ 1	☐ i	☐ $-2i$
☑ $-i$	☐ $2i$	☐ $i/2$

Exercise 4 (10 point)

(a) (5 point) A homogeneous second order differential equation is given by

$$y'' = 2y'.$$

Below there are given several functions where c_1 and c_2 are arbitrary real constants. Mark the expression which corresponds to the general solution of the differential equation.

☐ $y(t) = c_1e^{-t} + c_2e^t$	☑ $y(t) = c_1e^{2t} + c_2$
☐ $y(t) = c_1\cos(t) + c_2\sin(t)$	☐ $y(t) = c_1\sin(2t) + c_2\cos(2t)$
☐ $y(t) = c_1 + c_2t$	☐ $y(t) = c_1t^2 + c_2t$
☐ $y(t) = c_1 + c_2t^2$	☐ $y(t) = c_1 + c_2e^t$

(b) (5 point) Mark the solution $x(t)$ to the inhomogeneous differential equation

$$x''(t) = 2x'(t) + 1, \quad x(0) = 0, \quad x'(0) = 0,$$

among the following expressions:

☐ $x(t) = t^2$	☑ $x(t) = -\frac{t}{2} + \frac{1}{4}(e^{2t} - 1)$
☐ $x(t) = -4t^2 + 1$	☐ $x(t) = t - e^{2t}$
☐ $x(t) = t - te^{2t}$	☐ $x(t) = t - \sin(2t)$
☐ $x(t) = e^{2t} - 1 + t^2 - \frac{t}{2}$	☐ $x(t) = t - \cos(2t)$

Exercise 5 (8 point)

Answer whether the following statements are true or false.

(a) (2 point) The velocity vector and the unit tangent vector have always the same length.

☐ True ☑ False

(b) (2 point) When a point moves on a circle, the curvature is constant.

☑ True ☐ False

(c) (2 point) The product of any two real functions which are both differentiable at a point, is also differentiable at that point.

☑ True ☐ False

(d) (2 point) The function $f(x) = \sin(x)$ where $0 \leq x \leq 2\pi$, has an inverse function.

☐ True ☑ False

Exercise 6 (7 point)

A domain $\mathcal{R}$ in the plane can be represented with the help of the inequalities $4 \leq x^2 + y^2 \leq 9$ and $x \leq y$.

(a) (3 point) Which of the following inequalities show, that a point with coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$ belongs to $\mathcal{R}$?

☐ $r \geq 2, \quad 0 \leq \theta \leq \pi$ ☐ $4 \leq r \leq 9, \quad \theta = \pi/2$

☐ $4 \leq r \leq 9, \quad \pi/4 \leq \theta \leq 5\pi/4$ ☐ $r \leq 9$

☑ $2 \leq r \leq 3, \quad \pi/4 \leq \theta \leq 5\pi/4$ ☐ $2 \leq r \leq 3$

(b) (4 point) What is the area of the domain?

☐ $\pi/2$ ☐ $3\pi/2$ ☑ $5\pi/2$

☐ π ☐ 2π ☐ 3π

Exercise 7 (8 point)

A region $\mathcal{R}$ in the plane consists of all the points with coordinates (x, y) which satisfy

$$\sqrt{x^2 + y^2} \leq 2.$$

A function f is defined on $\mathcal{R}$ and is given by $f(x, y) = x^2 + 2y^2$.

(a) (4 point) Which of the following points is an inner critical point for f?

- ☐ $\langle 0, 1 \rangle$
- ☑ $\langle 0, 0 \rangle$
- ☐ $\langle 1, 0 \rangle$
- ☐ $\langle 1, 1 \rangle$
- ☐ $\langle 1, -1 \rangle$
- ☐ $\langle -1, -1 \rangle$

(b) (4 point) What is the maximum value of f?

- ☐ 2
- ☐ 4
- ☐ 6
- ☑ 8
- ☐ 10
- ☐ 12

Exercise 8 (12 point)

A surface $\mathcal{F}$ in space is determined by the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = x^2 + y^2 - 2z^2.$$

(a) (6 point) Which of the following equations correspond to the tangent plane to $\mathcal{F}$ at the point $P = (1, 1, 1)$?

- ☐ $3 = x + y + z$
- ☑ $2z = x + y$
- ☐ $z = -x + 2y$
- ☐ $z = -y + 2x$
- ☐ $z = 1$
- ☐ $0 = 3x + 2y - 5z$

(b) (6 point) From the equation $F(x, y, z) = 0$, what is the partial derivative $\partial z / \partial y$ at the point P?

- ☐ -1
- ☐ 0
- ☐ $-1/2$
- ☑ $1/2$
- ☐ $3/5$
- ☐ 4

Exercise 9 (12 point)

A function is given by

$$f(x,y) = \ln(e^x + y),$$

where $x > -1$ and $y > 0$.

(a) (2 point) Mark whether the following statement is true or false: $f(x,y)$ can never be less than zero.

- ☐ True
- ☑ False

(b) (2 point) Mark whether the following statement is true or false: $f(x,y)$ is always less than 100.

- ☐ True
- ☑ False

(c) (4 point) What is the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (0,1)$ and direction given by the unit vector $\mathbf{u} = \langle 1,0 \rangle$?

- ☐ 1
- ☐ 3
- ☐ 4
- ☐ 2
- ☐ $\sqrt{2}$
- ☑ $\frac{1}{2}$

(d) (4 point) Which of the following unit vectors point in the direction in which f grows fastest starting from P (the direction $\mathbf{v}$ for which $D_{\mathbf{v}}f(P)$ is maximal)?

- ☐ $\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$
- ☐ $\langle \frac{2\sqrt{5}}{5}, -\frac{1}{\sqrt{5}} \rangle$
- ☐ $\langle 0, -1 \rangle$
- ☐ $\langle -1, 0 \rangle$
- ☐ $\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \rangle$
- ☑ $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$
- ☐ $\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle$
- ☐ $\langle 1, 0 \rangle$
- ☐ $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

Exercise 10 (9 point)

A function is given by

$$f(x) = x^2 + \sin(2x)$$

for all real numbers x.

(a) (5 point) Mark the correct expression for $f'''(x)$ (i.e. the third order derivative of f)

☐ $\cos(2x)$ ☐ $8\cos(2x)$ ☐ $8\sin(2x)$
☐ $-2\cos(2x)$ ☑ $-8\cos(2x)$ ☐ $-8\sin(2x)$

(b) (4 point) Which one of the following expressions gives the third order Taylor polynomial for f when the developing point is $a = 0$?

☐ $1 + x + x^2 + x^3$ ☐ $-x + x^2 - x^3/6$ ☐ $2x + x^2$
☐ $1 + x^2/2 + x^3/6$ ☑ $2x + x^2 - 4x^3/3$ ☐ $2x + x^2 - 4x^3$

Exercise 11 (11 point)

A curve in the plane is given by

$$x(t) = \cos(2t),$$
$$y(t) = \sin(t)$$

for all real numbers t.

(a) (2 point) For which positive value of the parameter t does the curve get back to the point $P = (1, 0)$ for the first time? (Note that the curve is at P when $t = 0$.)

☐ $\pi/8$ ☐ $\pi/4$ ☐ $\pi/2$ ☑ π ☐ 2π

(b) (5 point) What is the curvature at P?

☐ 1 ☐ 2 ☐ 3 ☑ 4 ☐ 5

(c) (4 point) What is the value of the velocity when $t = \pi$?

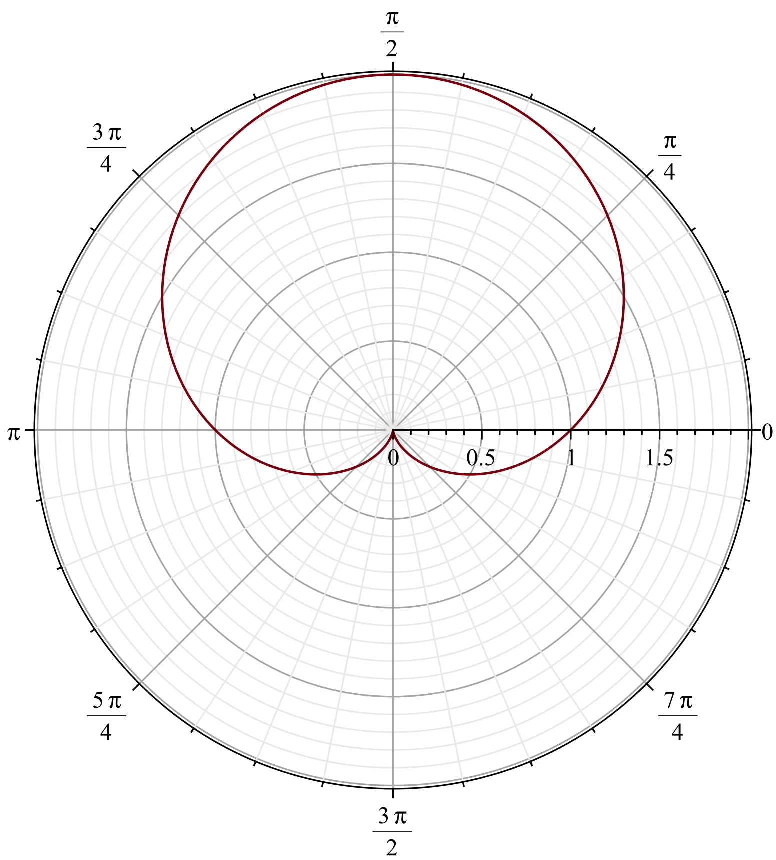
☐ 0 ☐ 2 ☐ 4
☑ 1 ☐ 3 ☐ 5

Opgave 12 (5 point)

The figure below shows the graph of a function

$$r = f(\theta), \quad 0 \le \theta \le 2\pi,$$

expressed in polar coordinates. The graph represents a cardioid.



One of the expressions below corresponds to the figure. Which one?

- ☐ $f(\theta) = 1 + \sin(2\theta)$
- ☐ $f(\theta) = 1 - \cos(\theta)$
- ☑ $f(\theta) = 1 + 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$
- ☐ $f(\theta) = \sin^2(\theta) - \cos(\theta)$
- ☐ $f(\theta) = \cos(\theta)\sin(\theta)$
- ☐ $f(\theta) = \frac{2-\sin(\theta)}{2}$