Retake exam in Calculus

19. august 2019

Exercise 1 (6 point)

A function is defined by

$$f(x, y, z) = \sin(x^2 + y^2 + z^2)$$

for real variables x, y and z.

(a) (3 point) The domain of definition of f consists of all the points (x, y, z) which satisfy

- ☐ $z = 0$
- ☐ $x^2 + y^2 + z^2 > 0$
- ☐ $x^2 + y^2 + z^2 < 0$
- ☑ all points are allowed
- ☐ $xyz \neq 0$
- ☐ none of the others

(b) (3 point) Which points (x, y, z) belong to the level surface determined by $f(x, y, z) = 2$?

- ☐ A sphere given by $x^2 + y^2 + z^2 = \sin^{-1}(2)$
- ☐ The xy-plane
- ☑ There are no such points
- ☐ A plane parallel with the xy-plane given by $z = \sin^{-1}(2)$
- ☐ none of the others

Exercise 2 (6 point)

A parametric curve in space is given by

$$\mathbf{r}(t) = \langle \sin(t),\ \cos(t),\ t \rangle$$

where the parameter t is an arbitrary real number.

(a) (2 point) What is the velocity vector of the curve?

- ☐ $\langle \sin(t), \cos(t), 1 \rangle$
- ☐ $\langle \cos(t), \sin(t), 1 \rangle$
- ☑ $\langle \cos(t), -\sin(t), 1 \rangle$
- ☐ $\langle \sin(t^2/2), \cos(t^2/2), t^2/2 \rangle$
- ☐ $\langle \sin(t^2/2), -\cos(t^2/2), t^2/2 \rangle$
- ☐ none of the others

(b) (2 point) Which of the following vectors is the acceleration vector for $t = 2\pi$?

- [x] $\langle 0,\ -1,\ 0\rangle$
- [ ] $\langle 1,\ 0,\ 0\rangle$
- [ ] $\langle 0,\ 1,\ 1/2\rangle$
- [ ] $\langle 0,\ 1,\ 0\rangle$
- [ ] $\langle 1,\ 1,\ 1\rangle$
- [ ] none of the others

(c) (1 point) What is the speed?

- [ ] $\sqrt{\sin(t)+\cos(t)+t}$
- [ ] 2
- [ ] $\sqrt{t+1}$
- [ ] $\sqrt{1+t^2}$
- [x] $\sqrt{2}$
- [ ] none of the others

(d) (1 point) What is the length of the curve from $t=0$ to $t=2$?

- [ ] 0
- [x] $2\sqrt{2}$
- [ ] $1/2$
- [ ] $\sqrt{2}$
- [ ] $4\sqrt{2}$
- [ ] none of the others

Exercise 3 (6 point)

Three complex numbers are given by

$$z_1 = 1+2i, \quad z_2 = 4-2i \quad \text{og} \quad z_3 = i^{100}.$$

(a) (2 point) What is $z_1 + z_2$ in polar form?

- [ ] 1
- [ ] $5e^{-i\pi/2}$
- [x] 5
- [ ] $5e^{i\pi/2}$
- [ ] $\sqrt{5}$
- [ ] none of the others

(b) (2 point) What is $\frac{2z_1}{z_2}$ in the standard form $a+ib$?

- [ ] $1+i$
- [ ] $-1-i$
- [ ] 5
- [x] i
- [ ] $-i$
- [ ] none of the others

(c) (2 point) What is the principal argument of z_3?

- [x] 0
- [ ] $\pi/2$
- [ ] π
- [ ] $\pi/4$
- [ ] $3\pi/4$
- [ ] none of the others

Hint to (c): The principal argument is a polar angle which belongs to the interval $]-\pi,\pi]$.

Exercise 4 (10 point)

(a) (5 point) A homogeneous second order differential equation is given by

$$y'' = -4y.$$

Below there are given a few functions where c_1 and c_2 are arbitrary real constants. Mark which expression represents the general solution to the differential equation.

- ☐ $y(t) = c_1 e^{-2t} + c_2 e^{2t}$
- ☐ $y(t) = c_1 e^{t} + c_2 t$
- ☐ $y(t) = c_1 \cos(t) + c_2 \sin(t)$
- ☑ $y(t) = c_1 \sin(2t) + c_2 \cos(2t)$
- ☐ $y(t) = (c_1 + c_2 t)e^{-2t}$
- ☐ $y(t) = (c_1 + c_2 t)e^{2t}$
- ☐ $y(t) = c_1 + c_2$
- ☐ none of the others

(b) (2 point) Which function $x_p(t)$ is a particular solution to the inhomogeneous differential equation

$$x''(t) = -4x(t) + 5e^t$$

among the following expressions:

- ☐ $x_p(t) = e^{2t}$
- ☐ $x_p(t) = e^{-t}$
- ☐ $x_p(t) = e^{-2t}$
- ☐ $x_p(t) = te^{t}$
- ☑ $x_p(t) = e^{t}$
- ☐ none of the others

(c) (3 point) Mark the solution $x(t)$ to the initial value problem

$$x''(t) = -4x(t) + 5e^t, \quad x(0) = 0, \quad x'(0) = 0,$$

among the following expressions:

- ☐ $x(t) = \sin(t) - t\cos(t)$
- ☑ $x(t) = -\sin(2t)/2 - \cos(2t) + e^t$
- ☐ $x(t) = -\sin(t) + te^t$
- ☐ $x(t) = t - te^{2t}$
- ☐ $x(t) = \sin(2t)/2 + \cos(2t) - e^t$
- ☐ none of the others

Exercise 5 (8 point)

Mark if the following statements are true or false:

(a) (2 point) $(\sqrt{2}+\sqrt{3})^2 = 5+2\sqrt{6}$.

☑ True ☐ False

(b) (2 point) $\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^2 = i$.

☑ True ☐ False

(c) (2 point) If $f(x) = \sin(x)$ and $g(t) = \cos(t)$ then $h(t) = f(g(t))$ is differentiable and $h'(t) = -\sin(t)\cos(\cos(t))$.

☑ True ☐ False

(d) (2 point) $e^{-\ln(x)} = \frac{1}{x}$.

☑ True ☐ False

Exercise 6 (7 point)

A function f is for $t \geq 0$ given by

$$f(t) = e^t \sin(t) + t^2.$$

(a) (3 point) Which of the following expressions correspond to $F(s) = \mathcal{L}(f)(s)$ for $s > 1$?

☐ $\frac{2}{s^4}+\frac{s-1}{(s-1)^2+1}$ ☐ $\frac{1}{s^3}+\frac{1}{(s+1)^2+1}$

☑ $\frac{2}{s^3}+\frac{1}{(s-1)^2+1}$ ☐ $\frac{2}{s^3}+\frac{2}{(s+1)^2+1}$

☐ $\frac{1}{s^4}+\frac{2}{(s+1)^2+1}$ ☐ none of the others

(b) (4 point) A function F is for $s > 1$ given by

$$F(s) = \frac{s}{s^2-1}.$$

Which of the following expressions correspond to $f(t) = \mathcal{L}^{-1}(F)(t)$ for $t \geq 0$ (inverse Laplace transform of F)?

☐ $f(t) = 1+t^2$ ☐ $f(t) = -e^t + e^{2t}$

☐ $f(t) = e^t + e^{-t}$ ☐ $f(t) = \sin(t)+\cos(t)$

☑ $f(t) = \frac{1}{2}e^{-t}(e^{2t}+1)$ ☐ none of the others

Opgave 7 (8 point)

A domain $\mathcal{R}$ in the plane consists of all the points with coordinates (x, y) which satisfy the inequality $x^2 + y^2 \leq 1$. The function f is defined on $\mathcal{R}$ and given by $f(x, y) = x + y$.

(a) (4 point) Which of the following points are inner critical points for f?

- ☐ $\langle 0, -1 \rangle$
- ☐ $\langle -1, 0 \rangle$
- ☐ $\langle 1, 0 \rangle$
- ☑ There are no inner critical points
- ☐ $\langle 0, 0 \rangle$
- ☐ none of the others

(b) (2 point) Mark whether the following statement is true or false:

The function f evaluated on the boundary of $\mathcal{R}$ takes the same values as the function $g(\theta) = \cos(\theta) + \sin(\theta)$, where $\theta \in [0, 2\pi]$.

- ☑ True
- ☐ False

(c) (2 point) What is the maximal value of f?

- ☐ 1
- ☐ 2
- ☐ 3
- ☑ $\sqrt{2}$
- ☐ $2\sqrt{2}$
- ☐ none of the others

Exercise 8 (12 point)

A surface $\mathcal{F}$ in space is determined by the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = x^4 + y^4 - 2z^2$$

(a) (3 point) Which of the following expressions gives the gradient vector ∇F?

- ☐ $\langle 4x, 4y, -4z \rangle$
- ☐ $\langle x^3, y^3, -2 \rangle$
- ☐ $\langle 4x^2, 4y^2, -4z^2 \rangle$
- ☐ $\langle 0, 0, 0 \rangle$
- ☑ $\langle 4x^3, 4y^3, -4z \rangle$
- ☐ none of the others

(b) (4 point) Which of the following equations represent the tangent plane to $\mathcal{F}$ at the point $P = (1, 1, 1)$?

- ☐ $0 = x + y + z$
- ☐ $z = x + y$
- ☐ $z = 2x - y$
- ☑ $x + y - z = 1$
- ☐ $z = y + 2x - 2$
- ☐ none of the others

(c) (5 point) From the equation $F(x, y, z) = 0$, what is the partial derivative $\partial z / \partial x$ at the point P?

- [ ] -1
- [ ] -2
- [ ] 2
- [ ] 0
- [x] 1
- [ ] none of the others

Exercise 9 (12 point)

A function is given by

$$f(x,y) = \sin(2x+y),$$

where $x \geq 0$ and $y \geq 0$.

(a) (2 point) Mark whether the following statement is true or false: $f(x,y)$ can never be equal to zero.

- [ ] True
- [x] False

(b) (2 point) Mark whether the following statement is true or false: $f(x,0)$ is an increasing function of x.

- [ ] True
- [x] False

(c) (4 point) What is the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (0,0)$ in the direction given by the unit vector $\mathbf{u} = \langle 0,1 \rangle$?

- [ ] 0
- [ ] 3
- [ ] 4
- [x] 1
- [ ] 2
- [ ] none of the others

(d) (4 point) Which of the following unit vectors point in the direction in which f grows fastest at the point P (the direction $\mathbf{v}$ for which $D_{\mathbf{v}}f(P)$ is largest)?

- [ ] $\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$
- [ ] $\langle \frac{2\sqrt{5}}{5}, -\frac{1}{\sqrt{5}} \rangle$
- [ ] $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$
- [ ] $\langle 1,0 \rangle$
- [x] $\langle \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \rangle$
- [ ] $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$
- [ ] $\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$
- [ ] $\langle 0,1 \rangle$
- [ ] none of the others

Exercise 10 (9 point)

A function is given by

$$f(x) = \cos(x^2)$$

for all real numbers x.

(a) (5 point) Mark the correct expression for $f''(x)$ (i.e. f twice differentiated)

- [ ] $-4\cos(x^2)$
- [ ] $2\cos(2x)$
- [ ] $-x^4\cos(x^2)$
- [ ] $2\sin(x^2) - x^2\cos(x^2)$
- [x] $-2\sin(x^2) - 4x^2\cos(x^2)$
- [ ] none of the others

(b) (4 point) Which of the following expressions represents the second order Taylor polynomial for f with the expansion point $x = 0$?

- ☐ $1 + x + x^2$
- ☐ $1 - x + x^2/2$
- ☐ $2x + x^2$
- ☐ $1 + x + x^2/2$
- ☑ 1
- ☐ none of the others

Exercise 11 (11 point)

A curve in the plane is given by

$$x(t) = t^2,$$
$$y(t) = \sin(t^3)$$

for all real numbers t.

(a) (2 point) For which value of the parameter t does the curve go through the origin?

- ☐ π
- ☐ 3π
- ☑ 0
- ☐ 2π
- ☐ 4π
- ☐ none of the others

(b) (4 point) What is the value of the speed when $t = 0$?

- ☑ 0
- ☐ $\sqrt{2}$
- ☐ $\sqrt{5}$
- ☐ 1
- ☐ $\sqrt{3}$
- ☐ none of the others

(c) (5 point) What is the acceleration vector at the origin?

- ☐ $\langle 0, 0 \rangle$
- ☐ $\langle 0, 1 \rangle$
- ☑ $\langle 2, 0 \rangle$
- ☐ $\langle 1, 0 \rangle$
- ☐ $\langle 1, 1 \rangle$
- ☐ none of the others

Exercise 12 (5 point)

Consider the following initial value problem

$$y'(x) = (x + 1)\, y(x), \quad y(0) = 1.$$

(a) (3 point) Assume that y solves the above equation and define

$$f(x) = \ln(y(x)).$$

Which initial value problem solves f?

- ☐ $f'(x) = x + 1, \quad f(0) = 1$
- ☐ $f'(x) = e^x, \quad f(0) = 0$
- ☐ $f'(x) = -x - 1, \quad f(0) = 1$
- ☐ $f'(x) = 1/(x + 1), \quad f(0) = 0$
- ☑ $f'(x) = x + 1, \quad f(0) = 0$
- ☐ none of the others

(b) (2 point) What is $y(x)$?

☐ $1+x$	☐ $1/(x+1)$	☐ $\ln(1+x+x^2/2)$
☐ $e^x - 1$	☑ $e^{x+x^2/2}$	☐ none of the others