For at finde den danske version af prøven, begynd i den modsatte ende!

Please disregard the Danish version on the back if you participate in this English version of the exam.

Exam in Calculus

First Year at the Technical Faculty for IT og Design,
the Faculty of Medicine and
the Faculty of Engineering and Science

June 15, 2018

This test consists of 8 numbered pages and 12 multiple choice problems. A number of points are assigned to each problem. The entire test consists of 100 points in total.

It is allowed to use books, notes, etc. It is **not allowed** to use any **electronic devices**.

Your answers must be marked on these sheets. In each subproblem you should only mark one of the listed choices. The evaluation is solely based on your marked answers on these sheets.

Remember to write your **full name** and **student number** below. Moreover, please mark the team that you participate in.

Good luck!

NAME: ________________________________

STUDENT NUMBER: ________________________________

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Problem 1 (6 points)

A function is given by

$$f(x, y) = e^{y-x^2}$$

with real variables x and y.

(a) (2 points) Mark the correct equation for the level curve $f(x, y) = 1$.

- ☐ $y = x^2 + 1$
- ☐ $y = x^2$
- ☐ $x = \sqrt{e^y - 1}$
- ☐ $y = x^2 + e$
- ☐ $y = \ln(x^2 + 1)$
- ☐ $x = \ln(1 - y)$

(b) (2 points) Which of the following vectors is parallel to $\nabla f(0, 0)$?

- ☐ $\langle 1, 0\rangle$
- ☐ $\langle 1, 4\rangle$
- ☐ $\langle 1, 1\rangle$
- ☐ $\langle 2, 1\rangle$
- ☐ $\langle 0, 2\rangle$

(c) (2 points) Which of the following functions agrees with the partial derivative f_{xy}?

- ☐ $-2xe^{y-x2}$
- ☐ $-2xe^{y-x^2}$
- ☐ $(1 - 2x)e^{y-x^2}$
- ☐ e^{y-x^2}
- ☐ 0
- ☐ e^{-2x}

Problem 2 (10 points)

A parametrized space curve is given by

$$\mathbf{r}(t) = \left\langle \tfrac{1}{3}t^3,\ \ t,\ \ \tfrac{\sqrt{2}}{2}t^2 \right\rangle$$

where the parameter t can take any real value.

(a) (4 points) Mark the correct expression for the speed $v(t)$.

- ☐ $t^2 + 1$
- ☐ $\sqrt{\frac{1}{9}t^6 + \frac{1}{2}t^4 + t^2}$
- ☐ $\sqrt{2}t(t + 1)$
- ☐ $\sqrt{t^4 + 2t^2} + 1$
- ☐ $t^2 + \sqrt{2}t + 1$
- ☐ $2t^2 + \sqrt{2}$

(b) (3 points) What is the arc-length of the curve from $t = 0$ to $t = 3$?

- ☐ 4
- ☐ 8
- ☐ 10
- ☐ 12
- ☐ 15

(c) (3 points) Which of the following vectors agrees with the curve's acceleration vector at $t = 2$?

- ☐ $\langle \frac{8}{3}, 2, 2\sqrt{2}\rangle$
- ☐ $\langle 2, 0, 0\rangle$
- ☐ $\langle 2, 0, \sqrt{2}\rangle$
- ☐ $\langle 4, 1, 2\sqrt{2}\rangle$
- ☐ $\langle 4, 0, \sqrt{2}\rangle$
- ☐ $\langle \frac{8}{3}, 2, \frac{\sqrt{2}}{2}\rangle$

Problem 3 (6 points)

Two complex numbers are given by

$$z_1 = 5e^{\frac{\pi}{3}i} \quad \text{and} \quad z_2 = 2i\left(\tfrac{3}{2} - 2i\right).$$

(a) (3 points) What is z_1 on standard form?

- ☐ $\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$
- ☐ $\frac{5\sqrt{3}}{2} + \frac{5}{2}i$
- ☐ $-\frac{5\sqrt{3}}{2} + \frac{5}{2}i$
- ☐ $\frac{5}{2} + \frac{5\sqrt{3}}{2}i$
- ☐ $5i$
- ☐ $10\sqrt{2} + 10\sqrt{3}i$

(b) (3 points) For all complex numbers w_1 and w_2 the equalities $|\overline{w_1}| = |w_1|$ and $|w_1 w_2| = |w_1||w_2|$ hold. What is $|2\, {z_1}^2\, \overline{z_2}|$?

☐ 50 ☐ 125 ☐ 250 ☐ 325 ☐ 380

Problem 4 (10 points)

A homogeneous second order differential equation is given by

$$2y'' + 3y' - 2y = 0.$$

(a) (5 points) Several functions are given below, where c_1 and c_2 are arbitrary real constants. Mark the function which agrees with the general solution of the differential equation.

- ☐ $y(t) = c_1 e^{-t} + c_2 e^{2t}$
- ☐ $y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{\frac{1}{2}t}$
- ☐ $y(t) = c_1 \cos(t) + c_2 \sin(t)$
- ☐ $y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{2t}$
- ☐ $y(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$
- ☐ $y(t) = c_1 e^{-2t} + c_2 e^{\frac{1}{2}t}$
- ☐ $y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{3t}$
- ☐ $y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

(b) (5 points) The function $x_p(t) = -t - 2$ is a particular solution of

$$2x'' + 3x' - 2x = 2t + 1.$$

Mark the unique solution of the initial value problem given by

$$2x'' + 3x' - 2x = 4t + 2, \;\; x(0) = 0, \;\; x'(0) = 0,$$

from the following list of functions.

- ☐ $x(t) = 2e^{\frac{1}{2}t} + 4t + 2$
- ☐ $x(t) = 3te^{-2t} - 2t - 4$
- ☐ $x(t) = 4e^{\frac{1}{2}t} - t - 2$
- ☐ $x(t) = 4e^{-2t} - 2t - 4$
- ☐ $x(t) = 4e^{\frac{1}{2}t} - 2t - 4$
- ☐ $x(t) = 4e^{-2t} + 4t + 2$
- ☐ $x(t) = 2e^{-2t} + 3e^{\frac{1}{2}t} - 2t - 4$
- ☐ $x(t) = \cos(t) - \sin(t) - t - 2$

Problem 5 (8 points)

A function is given by

$$f(x) = \sqrt{x^2 + 1}$$

with real variable x.

(a) (4 points) Mark the expression which agrees with $f''(x)$ (*hint: remember to use the chain rule and the product rule*).

☐ $\frac{1}{2\sqrt{x^2+1}}$	☐ $\frac{-1}{x^2+1}$	☐ $\frac{1}{(x^2+1)\sqrt{x^2+1}}$
☐ $\frac{1}{\sqrt{x^2+1}}$	☐ $\frac{x}{\sqrt{x^2+1}}$	☐ $\frac{1}{(x-1)(x+1)}$

(b) (4 points) Which of the following polynomials agrees with the second order Taylor polynomial of f about $x = 0$?

☐ $2 + x + x^2$	☐ $1 + x$	☐ $x - x^2$
☐ $\frac{1}{2} - x + 2x^2$	☐ $1 - \frac{1}{2}x + x^2$	☐ $\frac{1}{2} + \frac{1}{2}x^2$
☐ $1 + \frac{1}{2}x^2$	☐ $1 + \frac{3}{2}x^2$	☐ $\sqrt{2} + x^2$

Problem 6 (6 points)

A function f is for $t \geq 0$ defined by

$$f(t) = e^{2t}\sin(3t) + 2t^2 - 1.$$

Which of the following expressions agrees with $F(s) = \mathcal{L}(f)(s)$ for $s > 2$ (the Laplace transform of f)?

☐ $\frac{3}{(s-2)(s^2+9)} + \frac{4}{s^3} - \frac{1}{s}$	☐ $\frac{3}{(s-2)^2+9} + \frac{4-s^2}{s^3}$
☐ $\frac{s-2}{(s-2)^2+9} + \frac{4}{s^3} - \frac{1}{s}$	☐ $\frac{s-2}{(s-2)^2+9} + \frac{2-s^2}{s^3}$
☐ $\frac{3}{(s-2)^2+9} + \frac{3}{s^3} - \frac{1}{s}$	☐ $\frac{9}{(s-2)(s^2+9)} + \frac{1}{s^3} - \frac{1}{s}$

Problem 7 (6 points)

A function F is for $s > 2$ defined by

$$F(s) = \frac{4s-2}{(s-2)(s+4)}.$$

Which of the following expressions agrees with $f(t) = \mathcal{L}^{-1}(F)(t)$ for $t \geq 0$ (the inverse Laplace transform of F)?

- ☐ $\frac{1}{2}e^{-2t} + 4e^{4t}$
- ☐ $te^{-4t} + 2e^{2t}$
- ☐ $2e^{-4t} + 4e^{-2t}$
- ☐ $3e^{-4t} + 6e^{2t}$
- ☐ $\frac{1}{4}e^{2t} + \frac{4}{3}e^{-4t}$
- ☐ $e^{2t} + 3e^{-4t}$

Problem 8 (10 points)

A surface $\mathcal{F}$ is defined by the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = \cos(z) + 2z + xy - x^2.$$

(a) (5 points) Which of the listed equations determines the tangent plane of $\mathcal{F}$ at the point $P = (1, 0, 0)$?

- ☐ $1 = \frac{1}{2}x + y + z$
- ☐ $z = x - \frac{1}{2}y - 1$
- ☐ $z = 2$
- ☐ $y = 2 - x - 1$
- ☐ $1 = x - 2y + 2z$
- ☐ $0 = -2x + y + 2z$

(b) (5 points) From the equation $F(x, y, z) = 0$, what is the partial derivative $\partial z/\partial x$ evaluated at the point P?

- ☐ -2
- ☐ 0
- ☐ 1
- ☐ π
- ☐ 5
- ☐ 7

Problem 9 (14 points)

A function is given by

$$f(x,y) = \sqrt{2x^2 - xy + 2}$$

with real variables x and y.

(a) (2 points) The domain of f consists of all points (x,y) that satisfy

- ☐ $xy \leq 2x^2 + 2$
- ☐ $2x^2 - xy \leq 2$
- ☐ $y \leq 2x^2 + 2$
- ☐ $2x^2 - xy + 2 \leq 0$
- ☐ $x \geq \sqrt{\frac{1}{2}xy - 1}$
- ☐ $y \leq 2x + 2$

(b) (3 points) Which of the following points is a critical point of f?

- ☐ $(1,4)$
- ☐ $(5,1)$
- ☐ $(0,0)$
- ☐ $(0,4)$
- ☐ $(-1,-1)$
- ☐ $(2,3)$

(c) (4 points) What is the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (0,1)$ and in the direction given by the unit vector $\mathbf{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$?

- ☐ -1
- ☐ $-\frac{1}{4}$
- ☐ $\frac{1}{2}$
- ☐ $-\frac{1}{2}$
- ☐ $\frac{1}{4}$
- ☐ 1

(d) (5 points) Which of the following unit vectors points in the direction of steepest descend for f at the point P (the direction $\mathbf{v}$ for which $D_{\mathbf{v}}f(P)$ is as small as possible)?

- ☐ $\langle 0,1 \rangle$
- ☐ $\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$
- ☐ $\langle 1,0 \rangle$
- ☐ $\langle 0,-1 \rangle$
- ☐ $\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$
- ☐ $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$
- ☐ $\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$
- ☐ $\langle -1,0 \rangle$
- ☐ $\langle -\frac{3}{5}, \frac{4}{5} \rangle$

Problem 10 (8 points)

Consider the following first order differential equation

$$\frac{dy}{dx} = 3y + yx.$$

(a) (1 point) Mark whether the following statement is true or false: The differential equation is separable.

☐ True ☐ False

(b) (4 points) The differential equation has a solution which satisfies $y(0) = 2$. For this solution, what is $y(1)$?

☐ 2 ☐ $2e^{\frac{7}{2}}$ ☐ $3e^{\frac{7}{2}}$ ☐ $3e^{\frac{1}{2}}$ ☐ $2e^{\frac{1}{2}}$

(c) (3 points) The differential equation has *another* solution which satisfies $y'(0) = 2$. For this solution, what is $y(0)$?

☐ $\frac{1}{4}$ ☐ $\frac{1}{3}$ ☐ $\frac{2}{3}$ ☐ $\frac{3}{4}$ ☐ 2

Problem 11 (11 points)

A planar curve is given by

$$x = \cos(t) + t,$$
$$y = t^2 + 2t + 1.$$

(a) (2 points) For which value of the parameter t does the curve pass through the point $P = (1, 1)$?

☐ $-\pi$ ☐ -1 ☐ $-\frac{\pi}{4}$ ☐ 0 ☐ 1

(b) (4 points) What is the curvature of the curve at the point P?

☐ $\frac{1}{\sqrt{5}}$ ☐ $\frac{4}{5\sqrt{5}}$ ☐ $\frac{4}{5}$ ☐ $\frac{3}{5\sqrt{5}}$ ☐ $\frac{5}{3\sqrt{5}}$

(c) (5 points) Below are listed several different parametrizations of the tangent line to the curve at the point P. Which one of the listed parametrizations has constant speed equal to 1 for all values of t?

☐ $\langle 1 + \frac{e^t - e^{-t}}{2\sqrt{5}}, 1 + \frac{e^t - e^{-t}}{\sqrt{5}} \rangle$ ☐ $\langle 1 + \frac{1}{\sqrt{5}}t, 1 + \frac{2}{\sqrt{5}}t \rangle$

☐ $\langle 1 + \frac{1}{3\sqrt{5}}t^3, 1 + \frac{2}{3\sqrt{5}}t^3 \rangle$ ☐ $\langle 1 + \frac{1}{3\sqrt{5}}(t-1)^3, 1 + \frac{2}{3\sqrt{5}}(t-1)^3 \rangle$

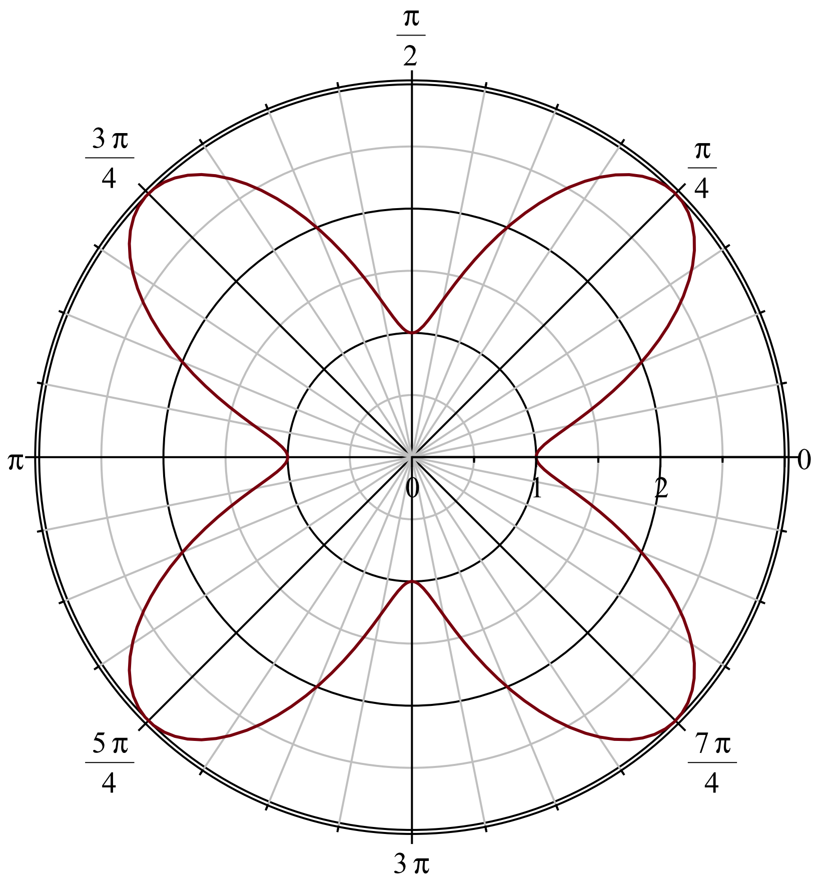
☐ $\langle 1 + t^3, 1 + 2t^3 \rangle$ ☐ $\langle 1 + \frac{3}{\sqrt{5}}t, 1 + \frac{6}{\sqrt{5}}t \rangle$

Problem 12 (5 points)

The figure below shows the graph of a function

$$r = f(\theta), \quad 0 \leq \theta \leq 2\pi,$$

in polar coordinates.



Which one the functions below gives rise to that graph?

- ☐ $f(\theta) = \sin(4\theta) - \cos(4\theta)$
- ☐ $f(\theta) = \sin(4\theta) - 2$
- ☐ $f(\theta) = 2 - \cos(4\theta)$
- ☐ $f(\theta) = \theta \sin(4\theta)$
- ☐ $f(\theta) = \cos(2\theta)$
- ☐ $f(\theta) = 2 + \sin(2\theta)$