

Exam in Calculus

14. june 2019

Exercise 1 (6 point)

A function is defined as

$$f(x, y, z) = 1 + \frac{z^2}{x^2 + y^2}$$

where x and y are real variables.

(a) (3 point) The domain of definition of f consists of all the points (x, y, z) which satisfy:

- $z \neq 0$
- $x + y > 0$
- $x^2 + y^2 \geq 1$
- The whole space without the z -axis
- $yx \neq 0$
- none of the others

(b) (3 point) What is the level surface defined by $f(x, y, z) = 1$?

- A sphere given by $x^2 + y^2 + z^2 = 1$
- The xy -plane without the origin
- A paraboloid given by $z = x^2 + y^2$
- A plane parallel with the xy -plane given by $z = 1$
- none of the others

Exercise 2 (6 point)

A parametric curve in space is given by

$$\mathbf{r}(t) = \langle \sin(2t), \cos(2t), 2t \rangle$$

where the parameter t can be any real number.

(a) (2 point) What is the velocity vector of the curve?

- $\langle 2 \sin(2t), 2 \cos(2t), 2 \rangle$
- $\langle 2 \cos(2t), -2 \sin(2t), 2t \rangle$
- $\langle 2 \cos(2t), 2 \sin(2t), 2 \rangle$
- none of the others

(b) (2 point) Which of the following vectors is the acceleration vector at $t = \pi$?

- $\langle 0, -1, 0 \rangle$ $\langle 0, -2, 0 \rangle$ $\langle 0, -2, 1 \rangle$
 $\langle 0, -4, 0 \rangle$ $\langle -4, 0, 1 \rangle$ none of the others

(c) (1 point) What is the velocity?

- \sqrt{t} $2\sqrt{2}$ $\sqrt{e^{i2t}}$
 $\sqrt{1+t^2}$ 4 none of the others

(d) (1 point) What is the length of the curve between $t = \pi$ and $t = 2\pi$?

- π $2\sqrt{2}\pi$ 5π
 2π 4π none of the others

Exercise 3 (6 point)

Three complex numbers are given by

$$z_1 = 1 + i, \quad z_2 = 2i^3 \quad \text{og} \quad z_3 = i^{10}.$$

(a) (2 point) What is $z_1 + z_2$ in polar form?

- 1 $-2e^{i\pi/4}$ $\sqrt{2}e^{-i\pi/4}$
 $2e^{-i\pi/4}$ $\sqrt{2}e^{\frac{5\pi}{4}i}$ none of the others

(b) (2 point) What is $\frac{z_1}{z_3}$ in the standard form $a + ib$?

- $1 - i$ $-1 - i$ $e^{-i\pi}$
 $1 + i$ $1 + i^9$ none of the others

(c) (2 point) What is the principal argument of z_1^5 ?

- 0 π $-3\pi/4$
 $\pi/4$ $3\pi/4$ none of the others

Hint to (c): The principal argument is a polar angle which belongs to the interval $]-\pi, \pi]$. A useful identity is $\arg(z^n) = n \arg(z)$.

Exercise 4 (10 point)

(a) (5 point) A homogeneous second order differential equation is given by

$$y'' = 2y' - y.$$

Below you may find a few expressions where c_1 and c_2 are arbitrary real constants. Mark the expression which gives the general solution to the differential equation.

- | | |
|---|---|
| <input type="checkbox"/> $y(t) = c_1 e^{-t} + c_2 e^t$ | <input type="checkbox"/> $y(t) = c_1 e^t + c_2 t$ |
| <input type="checkbox"/> $y(t) = c_1 \cos(t) + c_2 \sin(t)$ | <input type="checkbox"/> $y(t) = c_1 \sin(2t) + c_2 \cos(2t)$ |
| <input type="checkbox"/> $y(t) = c_1 + c_2 t$ | <input type="checkbox"/> $y(t) = (c_1 + c_2 t)e^t$ |
| <input type="checkbox"/> $y(t) = c_1 + c_2 t^2$ | <input type="checkbox"/> none of the others |

(b) (2 point) Which function $x_p(t)$ is a particular solution to the inhomogeneous differential equation

$$x''(t) = 2x'(t) - x(t) + 1$$

among the following expressions:

- | | |
|---|---|
| <input type="checkbox"/> $x_p(t) = t$ | <input type="checkbox"/> $x_p(t) = -t^2$ |
| <input type="checkbox"/> $x_p(t) = t + 1$ | <input type="checkbox"/> $x_p(t) = t - e^t$ |
| <input type="checkbox"/> $x_p(t) = 1$ | <input type="checkbox"/> none of the others |

(c) (3 point) Mark the solution $x(t)$ to the initial value problem

$$x''(t) = 2x'(t) - x(t) + 1, \quad x(0) = 0, \quad x'(0) = 1,$$

among the following expressions:

- | | |
|---|--|
| <input type="checkbox"/> $x(t) = e^t(2t + 1) - 1$ | <input type="checkbox"/> $x(t) = e^t t$ |
| <input type="checkbox"/> $x(t) = -4t^2$ | <input type="checkbox"/> $x(t) = t - e^{2t} + 1$ |
| <input type="checkbox"/> $x(t) = t - te^t$ | <input type="checkbox"/> $x(t) = 0$ |
| <input type="checkbox"/> $x(t) = e^t(2t - 1) + 1$ | <input type="checkbox"/> none of the others |

Exercise 5 (8 point)

Answer whether the following statements are true or false:

- (a) (2 point) The velocity vector of a curve can never have a zero length.
- True False
- (b) (2 point) When a points moves on a straight line, the curvature is infinitely large.
- True False
- (c) (2 point) If $f(x) = \cos(x)$ and $g(t) = e^t$, then $h(t) = f(g(t))$ is differentiable and $h'(t) = -\sin(e^t)$.
- True False
- (d) (2 point) The function $f(x) = \ln(x)$ where $0 < x < 1$ has an inverse function.
- True False

Exercise 6 (7 point)

A domain \mathcal{R} in the plane can be represented with the help of the inequality $x^2 + (y - 1)^2 \leq 1$.

- (a) (1 point) Which of the following curves describe the boundary of \mathcal{R} ?
- a parabola with is top point at $\langle 0, 1 \rangle$
- a circle with center at $\langle 0, 1 \rangle$ and radius 1
- a triangle with corners at the origin, $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$
- a square with side length 1 and center at i $\langle 0, 1 \rangle$
- none of the others
- (b) (2 point) Which of the following inequalities show that a point with coordinates $(x, y) = (r \cos(\theta), r \sin(\theta))$ belongs to \mathcal{R} ?
- $r \leq 2, \quad 0 \leq \theta \leq \pi$ $0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi$
- $0 \leq r \leq 2 \cos(\theta), \quad 0 \leq \theta \leq \pi$ $r \leq 1$
- $0 \leq r \leq 2 \sin(\theta), \quad 0 \leq \theta \leq \pi$ none of the others
- (c) (4 point) The domain \mathcal{R} models a plate with mass density $\delta(x, y) = y$. Which of the following integrals give the mass of the plate?

- $\int_0^\pi \int_0^{2\cos(\theta)} r^2 \cos(\theta) dr d\theta$
 $\int_0^\pi \int_0^2 r \sin(\theta) dr d\theta$
 $\int_0^\pi \int_0^{2\sin(\theta)} r^2 \sin(\theta) dr d\theta$
 $\int_0^\pi \int_0^{2\sin(\theta)} r \sin(\theta) dr d\theta$
 $\int_0^{2\pi} \int_0^1 r \sin(\theta) dr d\theta$
 none of the others

Exercise 7 (8 point)

A domain \mathcal{R} in the plane consists of all the points with coordinates (x, y) which obey the inequality $x^2 + (y - 1)^2 \leq 1$. The function f is defined on \mathcal{R} and is given by $f(x, y) = x^2$.

(a) (2 point) Which of the following points are inner critical points for f ?

- $\langle 0, -1 \rangle$
 $\langle t, 0 \rangle$ where $0 < t < 2$
 $\langle 1, 0 \rangle$
 $\langle t, t \rangle$ where $0 < t < 2$
 $\langle 0, t \rangle$ where $0 < t < 2$
 none of the others

(b) (4 point) Which of the following functions take the same values as f , when $\langle x, y \rangle$ belongs to the boundary of \mathcal{R} ?

- $g(y) = (y - 1)^2$ where $-1 \leq y \leq 0$
 $g(x) = 1 + x^2$ where $0 \leq x \leq 2$
 $g(y) = y^2$ where $0 \leq y \leq 2$
 $g(x) = x^2$ where $-1 \leq x \leq 1$
 $g(y) = 1 - y^2$ where $0 \leq y \leq 2$
 none of the others

(c) (2 point) What is the maximal value of f ?

- 1
 3
 5
 2
 4
 none of the others

Exercise 8 (12 point)

A surface \mathcal{F} in the space is determined by the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = 2y \sin(x) + y^2 - z^2.$$

(a) (3 point) Which of the following expressions correspond to the gradient vector ∇F ?

- $\langle -2y \cos(x), 2 \sin(x) + 2y, -2z \rangle$
 $\langle 2y \cos(x), 2 \sin(x) + 2y, -2z \rangle$
 $\langle 2y \cos(x) + 2 \sin(x), 2y, -2z \rangle$
 $\langle 0, 0, 0 \rangle$
 $\langle 2y \cos(x) + 2 \sin(x), 0, -2z \rangle$
 none of the others

(b) (3 point) Which of the following equations determine the tangent plane to \mathcal{F} at the point $P = (0, 1, 1)$?

- $2 = x + y + z$ $z = x + y$ $z = x - y$
 $2z = x + 2y$ $z = y + 2x$ none of the others

(c) (6 point) From the equation $F(x, y, z) = 0$, what is the partial derivative $\partial z / \partial x$ at the point P ?

- -1 -2 2
 0 1 none of the others

Exercise 9 (12 point)

A function is given by

$$f(x, y) = \sin(xy),$$

where $x \geq 0$ and $y \geq 0$.

(a) (2 point) Mark whether the following statement is true or false: $f(x, y)$ cannot be less than zero.

- True False

(b) (2 point) Mark whether the following statement is true or false: $f(x, y)$ can never be equal to $1/2$.

- True False

(c) (4 point) What is the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (0, 1)$ in the direction given by the unit vector $\mathbf{u} = \langle 0, 1 \rangle$?

- 0 3 4
 1 2 none of the others

(d) (4 point) Which of the following vectors has the same direction as the one in which f grows fastest at point P (the direction \mathbf{v} for which $D_{\mathbf{v}}f(P)$ is maximal)?

- $\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \rangle$ $\langle \frac{2\sqrt{5}}{5}, -\frac{1}{\sqrt{5}} \rangle$ $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$
 $\langle 1, 0 \rangle$ $\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \rangle$ $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$
 $\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle$ $\langle 0, 1 \rangle$ none of the others

Exercise 10 (9 point)

A function is given by

$$f(x) = \sin(x^2)$$

for all real numbers x .

(a) (5 point) Mark the correct expression for $f''(x)$ (which means f twice differentiated)

- $-4 \sin(x^2)$ $2 \sin(2x)$
 $-x^4 \sin(x^2)$ $2 \sin(x^2) - 4x^2 \cos(x^2)$
 $2 \cos(x^2) - 4x^2 \sin(x^2)$ none of the others

(b) (4 point) Which of the following expressions corresponds to the second order Taylor polynomial for f with the expansion point $a = 0$?

- $x + x^2$ $-x + x^2$ $2x + x^2$
 $1 + x + x^2/2$ x^2 none of the others

Exercise 11 (11 point)

A curve in the plane is given by

$$\begin{aligned}x(t) &= 2t + 1, \\y(t) &= \sin(t)\end{aligned}$$

for all real numbers t .

(a) (2 point) For which value of the parameter t does the curve go through the point $P = (1, 0)$?

- | | | |
|--------------------------------|---------------------------------|---|
| <input type="checkbox"/> 0 | <input type="checkbox"/> 2π | <input type="checkbox"/> 4π |
| <input type="checkbox"/> π | <input type="checkbox"/> 3π | <input type="checkbox"/> none of the others |

(b) (4 point) What is the value of the velocity when $t = 0$?

- | | | |
|----------------------------|-------------------------------------|---|
| <input type="checkbox"/> 0 | <input type="checkbox"/> $\sqrt{2}$ | <input type="checkbox"/> $\sqrt{5}$ |
| <input type="checkbox"/> 1 | <input type="checkbox"/> $\sqrt{3}$ | <input type="checkbox"/> none of the others |

(c) (5 point) What is the value of the curvature at P ?

- | | | |
|---------------------------------------|---|---|
| <input type="checkbox"/> 0 | <input type="checkbox"/> $1/\sqrt{27}$ | <input type="checkbox"/> $2/\sqrt{125}$ |
| <input type="checkbox"/> $1/\sqrt{8}$ | <input type="checkbox"/> $1/\sqrt{125}$ | <input type="checkbox"/> none of the others |

Exercise 12 (5 point)

Consider the following initial value problem

$$y'(x) = x y^2(x), \quad y(0) = 1.$$

(a) (3 point) Assume that y solves the above equation, and define

$$f(x) = \frac{1}{y(x)}.$$

Which equation does f satisfy?

- | | |
|---|--|
| <input type="checkbox"/> $f'(x) = x, \quad f(0) = 1$ | <input type="checkbox"/> $f'(x) = x^2, \quad f(0) = 0$ |
| <input type="checkbox"/> $f'(x) = -x, \quad f(0) = 1$ | <input type="checkbox"/> $f'(x) = x, \quad f(0) = 0$ |
| <input type="checkbox"/> $f'(x) = 1, \quad f(0) = 1$ | <input type="checkbox"/> none of the others |

(b) (2 point) What is $y(x)$?

- | | | |
|--------------------------------------|--|---|
| <input type="checkbox"/> $1 + x$ | <input type="checkbox"/> $2/(2 - x^2)$ | <input type="checkbox"/> $1/(1 - x^2)$ |
| <input type="checkbox"/> $1/(1 + x)$ | <input type="checkbox"/> $2/(2 + x^2)$ | <input type="checkbox"/> none of the others |