Exam in Calculus

14. june 2019

Exercise 1 (6 point)

A function is defined as

$$f(x,y,z) = 1 + \frac{z^2}{x^2+y^2}$$

where x and y are real variables.

(a) (3 point) The domain of definition of f consists of all the points (x,y,z) which satisfy:

- ☐ $z \neq 0$
- ☐ $x + y > 0$
- ☐ $x^2 + y^2 \geq 1$
- ☑ The whole space without the z-axis
- ☐ $yx \neq 0$
- ☐ none of the others

(b) (3 point) What is the level surface defined by $f(x,y,z) = 1$?

- ☐ A sphere given by $x^2 + y^2 + z^2 = 1$
- ☑ The xy-plane without the origin
- ☐ A paraboloid given by $z = x^2 + y^2$
- ☐ A plane parallel with the xy-plane given by $z = 1$
- ☐ none of the others

Exercise 2 (6 point)

A parametric curve in space is given by

$$\mathbf{r}(t) = \langle \sin(2t),\ \cos(2t),\ 2t \rangle$$

where the parameter t can be any real number.

(a) (2 point) What is the velocity vector of the curve?

- ☐ $\langle 2\sin(2t), 2\cos(2t), 2 \rangle$
- ☐ $\langle 2\cos(2t), 2\sin(2t), 2 \rangle$
- ☐ $\langle 2\cos(2t), -2\sin(2t), 2t \rangle$
- ☑ none of the others

(b) (2 point) Which of the following vectors is the acceleration vector at $t = \pi$?

- ☐ $\langle 0,\ -1,\ 0\rangle$
- ☐ $\langle 0,\ -2,\ 0\rangle$
- ☐ $\langle 0,\ -2,\ 1\rangle$
- ☑ $\langle 0,\ -4,\ 0\rangle$
- ☐ $\langle -4,\ 0,\ 1\rangle$
- ☐ none of the others

(c) (1 point) What is the velocity?

- ☐ $\sqrt{t}$
- ☑ $2\sqrt{2}$
- ☐ $\sqrt{e^{i2t}}$
- ☐ $\sqrt{1+t^2}$
- ☐ 4
- ☐ none of the others

(d) (1 point) What is the length of the curve between $t = \pi$ and $t = 2\pi$?

- ☐ π
- ☑ $2\sqrt{2}\pi$
- ☐ 5π
- ☐ 2π
- ☐ 4π
- ☐ none of the others

Exercise 3 (6 point)

Three complex numbers are given by

$$z_1 = 1 + i, \quad z_2 = 2i^3 \quad \text{og} \quad z_3 = i^{10}.$$

(a) (2 point) What is $z_1 + z_2$ in polar form?

- ☐ 1
- ☐ $-2e^{i\pi/4}$
- ☑ $\sqrt{2}e^{-i\pi/4}$
- ☐ $2e^{-i\pi/4}$
- ☐ $\sqrt{2}e^{\frac{5\pi}{4}i}$
- ☐ none of the others

(b) (2 point) What is $\frac{z_1}{z_3}$ in the standard form $a + ib$?

- ☐ $1 - i$
- ☑ $-1 - i$
- ☐ $e^{-i\pi}$
- ☐ $1 + i$
- ☐ $1 + i^9$
- ☐ none of the others

(c) (2 point) What is the principal argument of z_1^5?

- ☐ 0
- ☐ π
- ☑ $-3\pi/4$
- ☐ $\pi/4$
- ☐ $3\pi/4$
- ☐ none of the others

Hint to (c): The principal argument is a polar angle which belongs to the interval $]-\pi, \pi]$. A useful identity is $\arg(z^n) = n \arg(z)$.

Exercise 4 (10 point)

(a) (5 point) A homogeneous second order differential equation is given by

$$y'' = 2y' - y.$$

Below you may find a few expressions where c_1 and c_2 are arbitrary real constants. Mark the expression which gives the general solution to the differential equation.

- ☐ $y(t) = c_1 e^{-t} + c_2 e^t$
- ☐ $y(t) = c_1 e^t + c_2 t$
- ☐ $y(t) = c_1 \cos(t) + c_2 \sin(t)$
- ☐ $y(t) = c_1 \sin(2t) + c_2 \cos(2t)$
- ☐ $y(t) = c_1 + c_2 t$
- ☑ $y(t) = (c_1 + c_2 t)e^t$
- ☐ $y(t) = c_1 + c_2 t^2$
- ☐ none of the others

(b) (2 point) Which function $x_p(t)$ is a particular solution to the inhomogeneous differential equation

$$x''(t) = 2x'(t) - x(t) + 1$$

among the following expressions:

- ☐ $x_p(t) = t$
- ☐ $x_p(t) = -t^2$
- ☐ $x_p(t) = t + 1$
- ☐ $x_p(t) = t - e^t$
- ☑ $x_p(t) = 1$
- ☐ none of the others

(c) (3 point) Mark the solution $x(t)$ to the initial value problem

$$x''(t) = 2x'(t) - x(t) + 1, \quad x(0) = 0, \quad x'(0) = 1,$$

among the following expressions:

- ☐ $x(t) = e^t(2t + 1) - 1$
- ☐ $x(t) = e^t t$
- ☐ $x(t) = -4t^2$
- ☐ $x(t) = t - e^{2t} + 1$
- ☐ $x(t) = t - te^t$
- ☐ $x(t) = 0$
- ☑ $x(t) = e^t(2t - 1) + 1$
- ☐ none of the others

Exercise 5 (8 point)

Answer whether the following statements are true or false:

(a) (2 point) The velocity vector of a curve can never have a zero length.

☐ True ☑ False

(b) (2 point) When a points moves on a straight line, the curvature is infinitely large.

☐ True ☑ False

(c) (2 point) If $f(x) = \cos(x)$ and $g(t) = e^t$, then $h(t) = f(g(t))$ is differentiable and $h'(t) = -\sin(e^t)$.

☐ True ☑ False

(d) (2 point) The function $f(x) = \ln(x)$ where $0 < x < 1$ has an inverse function.

☑ True ☐ False

Exercise 6 (7 point)

A domain $\mathcal{R}$ in the plane can be represented with the help of the inequality $x^2 + (y-1)^2 \leq 1$.

(a) (1 point) Which of the following curves describe the boundary of $\mathcal{R}$?

- ☐ a parabola with is top point at $\langle 0, 1 \rangle$
- ☑ a circle with center at $\langle 0, 1 \rangle$ and radius 1
- ☐ a triangle with corners at the origin, $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$
- ☐ a square with side length 1 and center at i $\langle 0, 1 \rangle$
- ☐ none of the others

(b) (2 point) Which of the following inequalities show that a point with coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$ belongs to $\mathcal{R}$?

☐ $r \leq 2, \quad 0 \leq \theta \leq \pi$ ☐ $0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi$

☐ $0 \leq r \leq 2\cos(\theta), \quad 0 \leq \theta \leq \pi$ ☐ $r \leq 1$

☑ $0 \leq r \leq 2\sin(\theta), \quad 0 \leq \theta \leq \pi$ ☐ none of the others

(c) (4 point) The domain $\mathcal{R}$ models a plate with mass density $\delta(x, y) = y$. Which of the following integrals give the mass of the plate?

- ☐ $\int_0^{\pi}\int_0^{2\cos(\theta)} r^2\cos(\theta)dr\,d\theta$
- ☑ $\int_0^{\pi}\int_0^{2\sin(\theta)} r^2\sin(\theta)dr\,d\theta$
- ☐ $\int_0^{2\pi}\int_0^{1} r\sin(\theta)dr\,d\theta$
- ☐ $\int_0^{\pi}\int_0^{2} r\sin(\theta)dr\,d\theta$
- ☐ $\int_0^{\pi}\int_0^{2\sin(\theta)} r\sin(\theta)dr\,d\theta$
- ☐ none of the others

Exercise 7 (8 point)

A domain $\mathcal{R}$ in the plane consists of all the points with coordinates (x, y) which obey the inequality $x^2 + (y-1)^2 \leq 1$. The function f is defined on $\mathcal{R}$ and is given by $f(x, y) = x^2$.

(a) (2 point) Which of the following points are inner critical points for f?

- ☐ $\langle 0, -1\rangle$
- ☐ $\langle 1, 0\rangle$
- ☑ $\langle 0, t\rangle$ where $0 < t < 2$
- ☐ $\langle t, 0\rangle$ where $0 < t < 2$
- ☐ $\langle t, t\rangle$ where $0 < t < 2$
- ☐ none of the others

(b) (4 point) Which of the following functions take the same values as f, when $\langle x, y\rangle$ belongs to the boundary of $\mathcal{R}$?

- ☐ $g(y) = (y-1)^2$ where $-1 \leq y \leq 0$
- ☐ $g(y) = y^2$ where $0 \leq y \leq 2$
- ☐ $g(y) = 1 - y^2$ where $0 \leq y \leq 2$
- ☐ $g(x) = 1 + x^2$ where $0 \leq x \leq 2$
- ☑ $g(x) = x^2$ where $-1 \leq x \leq 1$
- ☐ none of the others

(c) (2 point) What is the maximal value of f?

- ☑ 1
- ☐ 2
- ☐ 3
- ☐ 4
- ☐ 5
- ☐ none of the others

Exercise 8 (12 point)

A surface $\mathcal{F}$ in the space is determined by the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = 2y\sin(x) + y^2 - z^2.$$

(a) (3 point) Which of the following expressions correspond to the gradient vector ∇F?

- ☐ $\langle -2y\cos(x), 2\sin(x) + 2y, -2z\rangle$
- ☐ $\langle 2y\cos(x) + 2\sin(x), 2y, -2z\rangle$
- ☐ $\langle 2y\cos(x) + 2\sin(x), 0, -2z\rangle$
- ☑ $\langle 2y\cos(x), 2\sin(x) + 2y, -2z\rangle$
- ☐ $\langle 0, 0, 0\rangle$
- ☐ none of the others

(b) (3 point) Which of the following equations determine the tangent plane to $\mathcal{F}$ at the point $P = (0, 1, 1)$?

- [ ] $2 = x + y + z$
- [x] $z = x + y$
- [ ] $z = x - y$
- [ ] $2z = x + 2y$
- [ ] $z = y + 2x$
- [ ] none of the others

(c) (6 point) From the equation $F(x,y,z) = 0$, what is the partial derivative $\partial z/\partial x$ at the point P?

- [ ] -1
- [ ] -2
- [ ] 2
- [ ] 0
- [x] 1
- [ ] none of the others

Exercise 9 (12 point)

A function is given by

$$f(x,y) = \sin(xy),$$

where $x \geq 0$ and $y \geq 0$.

(a) (2 point) Mark whether the following statement is true or false: $f(x,y)$ cannot be less than zero.

- [ ] True
- [x] False

(b) (2 point) Mark whether the following statement is true or false:$f(x,y)$ can never be equal to $1/2$.

- [ ] True
- [x] False

(c) (4 point) What is the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (0,1)$ in the direction given by the unit vector $\mathbf{u} = \langle 0,1\rangle$?

- [x] 0
- [ ] 3
- [ ] 4
- [ ] 1
- [ ] 2
- [ ] none of the others

(d) (4 point) Which of the following vectors has the same direction as the one in which f grows fastest at point P (the direction $\mathbf{v}$ for which $D_{\mathbf{v}}f(P)$ is maximal)?

- [ ] $\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\rangle$
- [ ] $\langle \frac{2\sqrt{5}}{5}, -\frac{1}{\sqrt{5}}\rangle$
- [ ] $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$
- [x] $\langle 1,0\rangle$
- [ ] $\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\rangle$
- [ ] $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$
- [ ] $\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\rangle$
- [ ] $\langle 0,1\rangle$
- [ ] none of the others

Exercise 10 (9 point)

A function is given by

$$f(x) = \sin(x^2)$$

for all real numbers x.

(a) (5 point) Mark the correct expression for $f''(x)$ (which means f twice differentiated)

- ☐ $-4\sin(x^2)$
- ☐ $-x^4\sin(x^2)$
- ☑ $2\cos(x^2) - 4x^2\sin(x^2)$
- ☐ $2\sin(2x)$
- ☐ $2\sin(x^2) - 4x^2\cos(x^2)$
- ☐ none of the others

(b) (4 point) Which of the following expressions corresponds to the second order Taylor polynomial for f with the expansion point $a = 0$?

- ☐ $x + x^2$
- ☐ $-x + x^2$
- ☐ $2x + x^2$
- ☐ $1 + x + x^2/2$
- ☑ x^2
- ☐ none of the others

Exercise 11 (11 point)

A curve in the plane is given by

$$x(t) = 2t + 1,$$
$$y(t) = \sin(t)$$

for all real numbers t.

(a) (2 point) For which value of the parameter t does the curve go through the point $P = (1, 0)$?

☑ 0	☐ 2π	☐ 4π
☐ π	☐ 3π	☐ none of the others

(b) (4 point) What is the value of the velocity when $t = 0$?

☐ 0	☐ $\sqrt{2}$	☑ $\sqrt{5}$
☐ 1	☐ $\sqrt{3}$	☐ none of the others

(c) (5 point) What is the value of the curvature at P?

☑ 0	☐ $1\sqrt{27}$	☐ $2/\sqrt{125}$
☐ $1/\sqrt{8}$	☐ $1/\sqrt{125}$	☐ none of the others

Exercise 12 (5 point)

Consider the following initial value problem

$$y'(x) = x\, y^2(x), \quad y(0) = 1.$$

(a) (3 point) Assume that y solves the above equation, and define

$$f(x) = \frac{1}{y(x)}.$$

Which equation does f satisfy?

☐ $f'(x) = x, \quad f(0) = 1$	☐ $f'(x) = x^2, \quad f(0) = 0$
☑ $f'(x) = -x, \quad f(0) = 1$	☐ $f'(x) = x, \quad f(0) = 0$
☐ $f'(x) = 1, \quad f(0) = 1$	☐ none of the others

(b) (2 point) What is $y(x)$?

☐ $1 + x$	☑ $2/(2 - x^2)$	☐ $1/(1 - x^2)$
☐ $1/(1 + x)$	☐ $2/(2 + x^2)$	☐ none of the others