Exam in Calculus

**First Year at the Technical Faculty for IT og Design,
the Faculty of Medicine and
the Faculty of Engineering and Science**

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Problem 1 (6 points)

A real valued function is given by

$$f(x, y) = \frac{x}{\sqrt{y - x^2}}$$

with real variables x and y.

(a) (3 points) The domain of f consists of all points (x, y) that satisfy

- ☐ $y = x^2$
- ☐ $y > x^2$
- ☐ $y \geq x^2$
- ☐ $y \leq x^2$
- ☐ $x, y \neq 0$
- ☐ none of the others

(b) (3 points) What is the level set $f(x, y) = 0$?

- ☐ The x-axis
- ☐ The y-axis
- ☐ The parabola $y = 2x^2$
- ☐ The positive part of the y-axis given by $y > 0$, $x = 0$
- ☐ none of the others

Problem 2 (6 points)

A parametric curve in space is given by

$$\mathbf{r}(t) = \left\langle t,\ \frac{1}{3}t^3,\ \frac{\sqrt{2}}{2}t^2 \right\rangle,$$

where the parameter t can be any real number.

(a) (2 points) What is the velocity vector of the curve?

- ☐ $\left\langle 1, 2t^2, \sqrt{2}t \right\rangle$
- ☐ $\left\langle 1, t^2, \sqrt{2}t \right\rangle$
- ☐ $\left\langle t^2, 1, \sqrt{2}t \right\rangle$
- ☐ $\left\langle t^2, \frac{1}{9}t^6, \frac{1}{2}t^4 \right\rangle$
- ☐ $\langle 1, t^4, 2t^2 \rangle$
- ☐ none of the others

(b) (2 points) Which of the following vectors is the acceleration vector at $t = -1$?

- ☐ $\langle 0,\ -1,\ \sqrt{2} \rangle$
- ☐ $\langle 1,\ -2,\ \sqrt{2} \rangle$
- ☐ $\langle 0,\ -2,\ \frac{\sqrt{2}}{2} \rangle$
- ☐ $\langle 0,\ 1,\ \sqrt{2} \rangle$
- ☐ $\langle 0,\ -2,\ \sqrt{2} \rangle$
- ☐ none of the others

(c) (1 points) What is the speed of the curve?

- ☐ $t^2 + 1$
- ☐ $\sqrt{1 + t^2 + \sqrt{2}t}$
- ☐ $t + 1$
- ☐ $\sqrt{1 + t^2}$
- ☐ $\sqrt{t + 1}$
- ☐ none of the others

(d) (1 points) What is the length of the curve from $t = 0$ to $t = 3$?

- ☐ 1
- ☐ 7
- ☐ 12
- ☐ 10
- ☐ $\frac{5}{2}$
- ☐ none of the others

Problem 3 (6 points)

Three complex numbers are given by

$$z_1 = 2 - 4i, \quad z_2 = -3 + i \quad \text{og} \quad z_3 = \pi + 7i.$$

(a) (2 points) What is the real part of e^{iz_3}?

- ☐ -7
- ☐ 0
- ☐ e^7
- ☐ π
- ☐ $-e^{-7}$
- ☐ none of the others

(b) (2 points) What is the imaginary part of e^{iz_3}?

- ☐ 0
- ☐ e^7
- ☐ $e^{-\pi}$
- ☐ 7
- ☐ $-\pi$
- ☐ none of the others

(c) (2 points) What is $z_1 - z_2$ in polar form?

- ☐ $25e^{-i\pi/4}$
- ☐ $2\sqrt{5}e^{-i\pi/4}$
- ☐ $25e^{i7\pi/4}$
- ☐ $5e^{i\pi/4}$
- ☐ $5\sqrt{2}e^{-i\pi/4}$
- ☐ none of the others

Problem 4 (10 point)

(a) (5 points) A homogeneous second order differential equation is given by

$$y'' - 3y' - 10y = 0.$$

Below there are given several functions where c_1 and c_2 are arbitrary real constants. Mark the expression which corresponds to the general solution of the differential equation.

- ☐ $y(t) = c_1 e^{-2t} + c_2 t e^{5t}$
- ☐ $y(t) = c_1 e^{5t} + c_2 t$
- ☐ $y(t) = e^{2t}(c_1 \cos(5t) + c_2 \sin(5t))$
- ☐ $y(t) = e^{5t}(c_1 \sin(2t) + c_2 \cos(2t))$
- ☐ $y(t) = (c_1 e^{5t} + c_2)e^{-2t}$
- ☐ $y(t) = (c_1 + c_2)e^{5t}$
- ☐ $y(t) = c_1 e^{5t} + c_2 e^{-2t}$
- ☐ none of the others

(b) (3 points) Mark the solution $x_p(t)$ to the inhomogeneous differential equation

$$x''(t) - 3x'(t) - 10x(t) = 20t.$$

among the following expressions:

- ☐ $x_p(t) = 20t$
- ☐ $x_p(t) = 2t$
- ☐ $x_p(t) = -\frac{1}{2} + 2t$
- ☐ $x_p(t) = \frac{3}{5} - 2t$
- ☐ $x_p(t) = -3 - 2t$
- ☐ none of the others

(c) (2 points) Mark the solution $x(t)$ to the initial value problem

$$x''(t) - 3x'(t) - 10x(t) = 20t, \quad x(0) = \frac{8}{5}, \quad x'(0) = 10,$$

among the following expressions:

- ☐ $x(t) = -e^{5t} - 3e^{-2t} + \frac{3}{5} - 2t$
- ☐ $x(t) = \frac{1}{5}e^{5t} - 2te^{-2t} - \frac{1}{2} + 2t$
- ☐ $x(t) = e^{5t} + 2e^{-2t} - \frac{1}{2} + 2t$
- ☐ $x(t) = (\frac{3}{5}e^{5t} - 1)e^{-2t} + 2t$
- ☐ $x(t) = 2e^{5t} - e^{-2t} + \frac{3}{5} - 2t$
- ☐ none of the others

Problem 5 (8 points)

Mark if the following statements are true or false:

(a) (2 points) The function $f(x, y) = \cos^2(e^{x+y})$ satisfies $f_{xy} = f_{yx}$.

- ☐ True
- ☐ False

(b) (2 points) $\sin^{-1}(\sin(\frac{5\pi}{2})) = \frac{5\pi}{2}$.

☐ True ☐ False

(c) (2 points) $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$.

☐ True ☐ False

(d) (2 points) $y' = x + 2y$ is a separable differential equation.

☐ True ☐ False

Problem 6 (7 points)

Let $\mathcal{R}$ be the region in the plane consisting of all points within and on the triangle with corners at the points $(-1, 0)$, $(-1, 1)$, $(1, 0)$ and let f be a function defined on $\mathcal{R}$ given by $f(x, y) = xy$.

(a) (2 points) Which of the following pairs of inequalities determine that a point with coordinates (x, y) belongs to$\mathcal{R}$?

☐ $0 \leq x \leq 1$, $-1 \leq y \leq 1$

☐ $-1 \leq x \leq 1, 0 \leq y \leq 1$

☐ $-1 \leq x \leq 1, 0 \leq y \leq -\frac{1}{2}x + \frac{1}{2}$

☐ $0 \leq y \leq 1$, $-1 \leq x \leq 2y + 1$

☐ none of the others

(b) (3 point) What is the correct formula that determines $\iint_{\mathcal{R}} f(x, y)\, dA$?

☐ $\displaystyle\int_0^1 \int_{-1}^1 xy\, dxdy$

☐ $\displaystyle\int_{-1}^1 \int_0^{-\frac{1}{2}x+\frac{1}{2}} xy\, dxdy$

☐ $\displaystyle\int_{-1}^{2y+1} \int_0^1 xy\, dydx$

☐ $\displaystyle\int_{-1}^1 x \int_0^{-\frac{1}{2}(x-1)} y\, dydx$

☐ $\displaystyle\int_0^1 \int_{-1}^{2y+1} xy\, dxdy$

☐ none of the others

(c) (2 points) Mark the correct value of the double integral $\iint_{\mathcal{R}} f(x, y)\, dA$:

☐ $-\frac{1}{6}$

☐ 0

☐ $\frac{1}{3}$

☐ $\frac{7}{6}$

☐ -1

☐ none of the others

Problem 7 (8 points)

A region $\mathcal{R}$ in the plane consists of all points with coordinates (x, y) that satisfy the inequalities: $x^2 + y^2 \leq 2$ and $y \geq 0$. A function f is defined on $\mathcal{R}$ and given by $f(x, y) = e^{-(x^2+y^2)}$.

(a) (3 points) Which of the following expressions describes $\mathcal{R}$?

- ☐ A disk with radius 1 and center at the origin
- ☐ A disk with radius 2 and center at the origin
- ☐ A disk with radius $\sqrt{2}$ and center at the origin
- ☐ An upper half disk with radius 2 and center at the origin
- ☐ An upper half disk with radius $\sqrt{2}$ and center at the origin
- ☐ none of the others

(b) (4 points) Mark the correct expression:

- ☐ $\langle 0, 1 \rangle$ is an inner critical point
- ☐ $\langle \frac{1}{2}, \frac{1}{2} \rangle$ is an inner critical point
- ☐ $\langle 0, 0 \rangle$ is an inner critical point
- ☐ $\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ is an inner critical point
- ☐ there are no inner critical points
- ☐ none of the others

(c) (1 points) What is the minimal value of f on $\mathcal{R}$?

- ☐ 1
- ☐ 0
- ☐ 2
- ☐ e^{-1}
- ☐ e^{-2}
- ☐ none of the others

Problem 8 (12 points)

A surface $\mathcal{F}$ in space is determined by the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = \sin(\pi x y z^{3/2})$$

(a) (4 points) Which of the following expressions gives the gradient vector $\nabla F(P)$ at the point $P = (-1, -1, 1)$?

- ☐ $\langle -\pi, -\pi, \pi \rangle$
- ☐ $\langle \pi, \pi, -\frac{3}{2}\pi \rangle$
- ☐ $\langle 1, 1, \frac{3}{2} \rangle$
- ☐ $\langle 1, 1, -1 \rangle$
- ☐ $\langle 0, 0, 0 \rangle$
- ☐ none of the others

(b) (4 points) Which of the following equations correspond to the tangent plane to $\mathcal{F}$ at the point $P = (-1, -1, 1)$

☐ $3z = 2x + 2y + 7$ ☐ $3z = 2x + y$ ☐ $-7z = 2x + 2y$

☐ $x + y - z = \frac{5}{2}$ ☐ $2z = 3y + 2x + 7$ ☐ none of the others

(c) (4 points) What is the partial derivative $\partial z/\partial x$ at the point $P = (-1, -1, 1)$?

☐ -1 ☐ $-\frac{2}{3}$ ☐ $\frac{2}{3}$

☐ 0 ☐ $-\frac{1}{3}$ ☐ none of the others

Problem 9 (12 points)

A function f is given by

$$f(x, y) = \sqrt{x^2 + y^2},$$

with real variables x and y. Let $\mathcal{R}$ be the region in the plane consisting of all points (x, y) which satisfy $x^2 + y^2 \leq 1$.

(a) (2 points) Mark whether the following statement is true or false: $\nabla f(0, 0) = \langle 0, 0 \rangle$.

☐ True ☐ False

(b) (2 points) Mark whether the following statement is true or false: $\iint_{\mathcal{R}} f(x, y)\, dA \geq 2$.

☐ True ☐ False

(c) (4 points) What is the directional derivative $D_{\mathbf{u}} f(P)$ at the point $P = (\sqrt{2}, \sqrt{2})$ and direction given by the unit vector $\mathbf{u} = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle$?

☐ 0 ☐ 3 ☐ 4

☐ 1 ☐ 2 ☐ none of the others

(d) (4 points) Which of the following functions agrees with the partial derivative f_{xy}?

☐ $\dfrac{2xy}{\sqrt{x^2 + y^2}}$ ☐ $\dfrac{2x + 2y}{\sqrt{x^2 + y^2}}$ ☐ $\dfrac{2xy}{(x^2 + y^2)^{3/2}}$

☐ $\dfrac{-xy}{\sqrt{x^2 + y^2}^3}$ ☐ $\dfrac{2x + 2y}{(x^2 + y^2)^{3/2}}$ ☐ $\dfrac{-4xy}{(x^2 + y^2)^{3/2}}$

☐ none of the others

Problem 10 (9 points)

A function is given by

$$f(x) = \frac{2\ln(x)}{x}$$

for all real numbers $x > 0$.

(a) (5 points) Mark the expression which agrees with $f''(x)$ (i.e. f twice differentiated)

- ☐ $\frac{6+4\ln(x)}{x^3}$
- ☐ $\frac{-2}{x^2}$
- ☐ $-\frac{1}{x^2}$
- ☐ $\frac{4\ln(x)-6}{x^3}$
- ☐ $\frac{2-2\ln(x)}{x}$
- ☐ none of the others

(b) (4 points) Which of the following expressions represents the second order Taylor polynomial for f with the expansion point $a = 1$?

- ☐ $2(x-1) + 3(x-1)^2$
- ☐ $-3x^2 + 2x$
- ☐ $-3x^2 + 6x - 2$
- ☐ $-3x^2 + 8x - 5$
- ☐ $-3x^2 + 2x + 1$
- ☐ none of the others

Problem 11 (11 points)

A curve in the plane is given by

$$x(t) = \cos(t),$$
$$y(t) = \sin(2t)$$

for all real numbers $t \geq 0$.

(a) (2 points) What is the minimal value of the parameter t for which the curve passes through the point $(-1, 0)$?

- ☐ π
- ☐ 0
- ☐ $\frac{3\pi}{2}$
- ☐ $\frac{\pi}{2}$
- ☐ 2π
- ☐ none of the others

(b) (4 points) What is the curvature when $t = \frac{\pi}{2}$?

- ☐ 0
- ☐ $\frac{1}{\sqrt{2}}$
- ☐ $\frac{3}{8}$
- ☐ $\frac{1}{2}$
- ☐ 1
- ☐ none of the others

(c) (5 points) What is the curvature at $(-1, 0)$?

☐ 1	☐ $\frac{1}{4}$	☐ $\frac{\sqrt{2}}{2}$
☐ 0	☐ $\frac{1}{2}$	☐ none of the others

Problem 12 (5 points)

Consider the following first order differential equation

$$y'(x) + \frac{1}{x}y(x) = -x^2,$$

for all $x > 0$.

(a) (3 points) What is the general solution of the differential equation?

☐ $-\frac{1}{4}x^3 + c\frac{1}{x}$	☐ $-x^2 - cx$
☐ $\frac{1}{4}x^5 - cx$	☐ $c\frac{1}{x}$
☐ $-x^2 + c\frac{1}{x}$	☐ none of the others

(b) (2 points) What is the solution to the initial value problem

$$y'(x) + \frac{1}{x}y(x) = -x^2, \quad y(1) = \frac{1}{2},$$

for all $x > 0$.

☐ $-\frac{1}{4}x^3$	☐ $\frac{1}{4}x^5 - \frac{1}{4}x$	☐ $\frac{1}{2x}$
☐ $-\frac{1}{4}x^3 + \frac{3}{4x}$	☐ $-x^2 - \frac{3}{2x}$	☐ none of the others