

Exam in Calculus

First Year at the Technical Faculty for IT og Design,
the Faculty of Medicine and
the Faculty of Engineering and Science

January 14, 2020

Problem 1 (6 points)

A real valued function is given by

$$f(x, y) = \frac{x}{\sqrt{y - x^2}}$$

with real variables x and y .

(a) (3 points) The domain of f consists of all points (x, y) that satisfy

- | | |
|---|---|
| <input type="checkbox"/> $y = x^2$ | <input type="checkbox"/> $y \leq x^2$ |
| <input checked="" type="checkbox"/> $y > x^2$ | <input type="checkbox"/> $x, y \neq 0$ |
| <input type="checkbox"/> $y \geq x^2$ | <input type="checkbox"/> none of the others |

(b) (3 points) What is the level set $f(x, y) = 0$?

- The x -axis
- The y -axis
- The parabola $y = 2x^2$
- The positive part of the y -axis given by $y > 0, x = 0$
- none of the others

Problem 2 (6 points)

A parametric curve in space is given by

$$\mathbf{r}(t) = \left\langle t, \frac{1}{3}t^3, \frac{\sqrt{2}}{2}t^2 \right\rangle,$$

where the parameter t can be any real number.

(a) (2 points) What is the velocity vector of the curve?

- | | |
|---|--|
| <input type="checkbox"/> $\langle 1, 2t^2, \sqrt{2}t \rangle$ | <input checked="" type="checkbox"/> $\langle 1, t^2, \sqrt{2}t \rangle$ |
| <input type="checkbox"/> $\langle t^2, 1, \sqrt{2}t \rangle$ | <input type="checkbox"/> $\langle t^2, \frac{1}{9}t^6, \frac{1}{2}t^4 \rangle$ |
| <input type="checkbox"/> $\langle 1, t^4, 2t^2 \rangle$ | <input type="checkbox"/> none of the others |

(b) (2 points) Which of the following vectors is the acceleration vector at $t = -1$?

- | | | |
|--|---|--|
| <input type="checkbox"/> $\langle 0, -1, \sqrt{2} \rangle$ | <input type="checkbox"/> $\langle 1, -2, \sqrt{2} \rangle$ | <input type="checkbox"/> $\langle 0, -2, \frac{\sqrt{2}}{2} \rangle$ |
| <input type="checkbox"/> $\langle 0, 1, \sqrt{2} \rangle$ | <input checked="" type="checkbox"/> $\langle 0, -2, \sqrt{2} \rangle$ | <input type="checkbox"/> none of the others |

(c) (1 points) What is the speed of the curve?

- | | | |
|---|---|---|
| <input checked="" type="checkbox"/> $t^2 + 1$ | <input type="checkbox"/> $\sqrt{1 + t^2 + \sqrt{2}t}$ | <input type="checkbox"/> $t + 1$ |
| <input type="checkbox"/> $\sqrt{1 + t^2}$ | <input type="checkbox"/> $\sqrt{t + 1}$ | <input type="checkbox"/> none of the others |

(d) (1 points) What is the length of the curve from $t = 0$ to $t = 3$?

- | | | |
|-----------------------------|--|---|
| <input type="checkbox"/> 1 | <input type="checkbox"/> 7 | <input checked="" type="checkbox"/> 12 |
| <input type="checkbox"/> 10 | <input type="checkbox"/> $\frac{5}{2}$ | <input type="checkbox"/> none of the others |

Problem 3 (6 points)

Three complex numbers are given by

$$z_1 = 2 - 4i, \quad z_2 = -3 + i \quad \text{og} \quad z_3 = \pi + 7i.$$

(a) (2 points) What is the real part of e^{iz_3} ?

- | | | |
|--------------------------------|---|---|
| <input type="checkbox"/> -7 | <input type="checkbox"/> 0 | <input type="checkbox"/> e^7 |
| <input type="checkbox"/> π | <input checked="" type="checkbox"/> $-e^{-7}$ | <input type="checkbox"/> none of the others |

(b) (2 points) What is the imaginary part of e^{iz_3} ?

- | | | |
|---------------------------------------|---------------------------------|---|
| <input checked="" type="checkbox"/> 0 | <input type="checkbox"/> e^7 | <input type="checkbox"/> $e^{-\pi}$ |
| <input type="checkbox"/> 7 | <input type="checkbox"/> $-\pi$ | <input type="checkbox"/> none of the others |

(c) (2 points) What is $z_1 - z_2$ in polar form?

- | | | |
|--|--|---|
| <input type="checkbox"/> $25e^{-i\pi/4}$ | <input type="checkbox"/> $2\sqrt{5}e^{-i\pi/4}$ | <input type="checkbox"/> $25e^{i7\pi/4}$ |
| <input type="checkbox"/> $5e^{i\pi/4}$ | <input checked="" type="checkbox"/> $5\sqrt{2}e^{-i\pi/4}$ | <input type="checkbox"/> none of the others |

Problem 4 (10 point)

- (a) (5 points) A homogeneous second order differential equation is given by

$$y'' - 3y' - 10y = 0.$$

Below there are given several functions where c_1 and c_2 are arbitrary real constants. Mark the expression which corresponds to the general solution of the differential equation.

- | | |
|---|---|
| <input type="checkbox"/> $y(t) = c_1e^{-2t} + c_2te^{5t}$ | <input type="checkbox"/> $y(t) = c_1e^{5t} + c_2t$ |
| <input type="checkbox"/> $y(t) = e^{2t}(c_1 \cos(5t) + c_2 \sin(5t))$ | <input type="checkbox"/> $y(t) = e^{5t}(c_1 \sin(2t) + c_2 \cos(2t))$ |
| <input type="checkbox"/> $y(t) = (c_1e^{5t} + c_2)e^{-2t}$ | <input type="checkbox"/> $y(t) = (c_1 + c_2)e^{5t}$ |
| <input checked="" type="checkbox"/> $y(t) = c_1e^{5t} + c_2e^{-2t}$ | <input type="checkbox"/> none of the others |

- (b) (3 points) Mark the solution $x_p(t)$ to the inhomogeneous differential equation

$$x''(t) - 3x'(t) - 10x(t) = 20t.$$

among the following expressions:

- | | |
|---|---|
| <input type="checkbox"/> $x_p(t) = 20t$ | <input type="checkbox"/> $x_p(t) = 2t$ |
| <input type="checkbox"/> $x_p(t) = -\frac{1}{2} + 2t$ | <input checked="" type="checkbox"/> $x_p(t) = \frac{3}{5} - 2t$ |
| <input type="checkbox"/> $x_p(t) = -3 - 2t$ | <input type="checkbox"/> none of the others |

- (c) (2 points) Mark the solution $x(t)$ to the initial value problem

$$x''(t) - 3x'(t) - 10x(t) = 20t, \quad x(0) = \frac{8}{5}, \quad x'(0) = 10,$$

among the following expressions:

- | | |
|---|--|
| <input type="checkbox"/> $x(t) = -e^{5t} - 3e^{-2t} + \frac{3}{5} - 2t$ | <input type="checkbox"/> $x(t) = \frac{1}{5}e^{5t} - 2te^{-2t} - \frac{1}{2} + 2t$ |
| <input type="checkbox"/> $x(t) = e^{5t} + 2e^{-2t} - \frac{1}{2} + 2t$ | <input type="checkbox"/> $x(t) = (\frac{3}{5}e^{5t} - 1)e^{-2t} + 2t$ |
| <input checked="" type="checkbox"/> $x(t) = 2e^{5t} - e^{-2t} + \frac{3}{5} - 2t$ | <input type="checkbox"/> none of the others |

Problem 5 (8 points)

Mark if the following statements are true or false:

- (a) (2 points) The function $f(x, y) = \cos^2(e^{x+y})$ satisfies $f_{xy} = f_{yx}$.

- | | |
|--|--------------------------------|
| <input checked="" type="checkbox"/> True | <input type="checkbox"/> False |
|--|--------------------------------|

(b) (2 points) $\sin^{-1}(\sin(\frac{5\pi}{2})) = \frac{5\pi}{2}$.

True

False

(c) (2 points) $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$.

True

False

(d) (2 points) $y' = x + 2y$ is a separable differential equation.

True

False

Problem 6 (7 points)

Let \mathcal{R} be the region in the plane consisting of all points within and on the triangle with corners at the points $(-1,0)$, $(-1,1)$, $(1,0)$ and let f be a function defined on \mathcal{R} given by $f(x,y) = xy$.

(a) (2 points) Which of the following pairs of inequalities determine that a point with coordinates (x,y) belongs to \mathcal{R} ?

$0 \leq x \leq 1, -1 \leq y \leq 1$

$-1 \leq x \leq 1, 0 \leq y \leq 1$

$-1 \leq x \leq 1, 0 \leq y \leq -\frac{1}{2}x + \frac{1}{2}$

$0 \leq y \leq 1, -1 \leq x \leq 2y + 1$

none of the others

(b) (3 point) What is the correct formula that determines $\iint_{\mathcal{R}} f(x,y) dA$?

$\int_0^1 \int_{-1}^1 xy dx dy$

$\int_{-1}^1 \int_0^{-\frac{1}{2}x + \frac{1}{2}} xy dx dy$

$\int_{-1}^{2y+1} \int_0^1 xy dy dx$

$\int_{-1}^1 x \int_0^{-\frac{1}{2}(x-1)} y dy dx$

$\int_0^1 \int_{-1}^{2y+1} xy dx dy$

none of the others

(c) (2 points) Mark the correct value of the double integral $\iint_{\mathcal{R}} f(x,y) dA$:

$-\frac{1}{6}$

0

$\frac{1}{3}$

$\frac{7}{6}$

-1

none of the others

Problem 7 (8 points)

A region \mathcal{R} in the plane consists of all points with coordinates (x, y) that satisfy the inequalities: $x^2 + y^2 \leq 2$ and $y \geq 0$. A function f is defined on \mathcal{R} and given by $f(x, y) = e^{-(x^2+y^2)}$.

(a) (3 points) Which of the following expressions describes \mathcal{R} ?

- A disk with radius 1 and center at the origin
- A disk with radius 2 and center at the origin
- A disk with radius $\sqrt{2}$ and center at the origin
- An upper half disk with radius 2 and center at the origin
- An upper half disk with radius $\sqrt{2}$ and center at the origin
- none of the others

(b) (4 points) Mark the correct expression:

- $\langle 0, 1 \rangle$ is an inner critical point
- $\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ is an inner critical point
- $\langle \frac{1}{2}, \frac{1}{2} \rangle$ is an inner critical point
- there are no inner critical points
- $\langle 0, 0 \rangle$ is an inner critical point
- none of the others

(c) (1 points) What is the minimal value of f on \mathcal{R} ?

- 1
- 2
- e^{-2}
- 0
- e^{-1}
- none of the others

Problem 8 (12 points)

A surface \mathcal{F} in space is determined by the equation $F(x, y, z) = 0$, where

$$F(x, y, z) = \sin(\pi x y z^{3/2})$$

(a) (4 points) Which of the following expressions gives the gradient vector $\nabla F(P)$ at the point $P = (-1, -1, 1)$?

- $\langle -\pi, -\pi, \pi \rangle$
- $\langle \pi, \pi, -\frac{3}{2}\pi \rangle$
- $\langle 1, 1, \frac{3}{2} \rangle$
- $\langle 1, 1, -1 \rangle$
- $\langle 0, 0, 0 \rangle$
- none of the others

(b) (4 points) Which of the following equations correspond to the tangent plane to \mathcal{F} at the point $P = (-1, -1, 1)$

- $3z = 2x + 2y + 7$ $3z = 2x + y$ $-7z = 2x + 2y$
 $x + y - z = \frac{5}{2}$ $2z = 3y + 2x + 7$ none of the others

(c) (4 points) What is the partial derivative $\partial z / \partial x$ at the point $P = (-1, -1, 1)$?

- -1 $-\frac{2}{3}$ $\frac{2}{3}$
 0 $-\frac{1}{3}$ none of the others

Problem 9 (12 points)

A function f is given by

$$f(x, y) = \sqrt{x^2 + y^2},$$

with real variables x and y . Let \mathcal{R} be the region in the plane consisting of all points (x, y) which satisfy $x^2 + y^2 \leq 1$.

(a) (2 points) Mark whether the following statement is true or false: $\nabla f(0, 0) = \langle 0, 0 \rangle$.

- True False

(b) (2 points) Mark whether the following statement is true or false: $\iint_{\mathcal{R}} f(x, y) dA \geq 2$.

- True False

(c) (4 points) What is the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (\sqrt{2}, \sqrt{2})$ and direction given by the unit vector $\mathbf{u} = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle$?

- 0 3 4
 1 2 none of the others

(d) (4 points) Which of the following functions agrees with the partial derivative f_{xy} ?

- $\frac{2xy}{\sqrt{x^2 + y^2}}$ $\frac{2x + 2y}{\sqrt{x^2 + y^2}}$ $\frac{2xy}{(x^2 + y^2)^{3/2}}$
 $\frac{-xy}{\sqrt{x^2 + y^2}^3}$ $\frac{2x + 2y}{(x^2 + y^2)^{3/2}}$ $\frac{-4xy}{(x^2 + y^2)^{3/2}}$
 none of the others

Problem 10 (9 points)

A function is given by

$$f(x) = \frac{2\ln(x)}{x}$$

for all real numbers $x > 0$.

(a) (5 points) Mark the expression which agrees with $f'''(x)$ (i.e. f twice differentiated)

$\frac{6+4\ln(x)}{x^3}$

$\frac{-2}{x^2}$

$-\frac{1}{x^2}$

$\frac{4\ln(x)-6}{x^3}$

$\frac{2-2\ln(x)}{x}$

 none of the others

(b) (4 points) Which of the following expressions represents the second order Taylor polynomial for f with the expansion point $a = 1$?

$2(x-1) + 3(x-1)^2$

$-3x^2 + 2x$

$-3x^2 + 6x - 2$

$-3x^2 + 8x - 5$

$-3x^2 + 2x + 1$

 none of the others

Problem 11 (11 points)

A curve in the plane is given by

$$x(t) = \cos(t),$$

$$y(t) = \sin(2t)$$

for all real numbers $t \geq 0$.

(a) (2 points) What is the minimal value of the parameter t for which the curve passes through the point $(-1, 0)$?

π

0

$\frac{3\pi}{2}$

$\frac{\pi}{2}$

2π

 none of the others

(b) (4 points) What is the curvature when $t = \frac{\pi}{2}$?

0

$\frac{1}{\sqrt{2}}$

$\frac{3}{8}$

$\frac{1}{2}$

1

 none of the others

(c) (5 points) What is the curvature at $(-1, 0)$?

1

$\frac{1}{4}$

$\frac{\sqrt{2}}{2}$

0

$\frac{1}{2}$

none of the others

Problem 12 (5 points)

Consider the following first order differential equation

$$y'(x) + \frac{1}{x}y(x) = -x^2,$$

for all $x > 0$.

(a) (3 points) What is the general solution of the differential equation?

$-\frac{1}{4}x^3 + c\frac{1}{x}$

$-x^2 - cx$

$\frac{1}{4}x^5 - cx$

$c\frac{1}{x}$

$-x^2 + c\frac{1}{x}$

none of the others

(b) (2 points) What is the solution to the initial value problem

$$y'(x) + \frac{1}{x}y(x) = -x^2, \quad y(1) = \frac{1}{2},$$

for all $x > 0$.

$-\frac{1}{4}x^3$

$\frac{1}{4}x^5 - \frac{1}{4}x$

$\frac{1}{2x}$

$-\frac{1}{4}x^3 + \frac{3}{4x}$

$-x^2 - \frac{3}{2x}$

none of the others