Exam in Calculus

First Year at the Faculty of Engineering and Science
and the Technical Faculty of IT and Design

5 January 2018

The present exam set consists of 8 numbered pages with 12 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use books, notes etc. It **is not** allowed to use electronic devices.

Your answers must be marked directly in this exam set. The grading of the exam is based solely on these marks. Remember to write your **full name** and **student number** below.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Problem 1 (9 points)

(a) (5 points). A second order differential equation is given by

$$y'' + 4y' + 29y = 0.$$

Below is a list of function expressions containing two arbitrary constants c_1 and c_2. Mark the expression which constitute the general solution of the differential equation.

- ☐ $y(t) = c_1 e^{3t} + c_2 e^{5t}$
- ☐ $y(t) = c_1 e^{2t} + c_2 e^{5t}$
- ☐ $y(t) = c_1 e^{-2t} + c_2 e^{3t}$
- ☐ $y(t) = c_1 e^{2t} + c_2 t e^{2t}$
- ☐ $y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$
- ☐ $y(t) = c_1 e^{-2t} \cos(5t) + c_2 e^{-2t} \sin(5t)$
- ☐ $y(t) = c_1 e^{-t} \cos(4t) + c_2 e^{-t} \sin(4t)$
- ☐ $y(t) = c_1 e^{2t} \cos(3t) + c_2 e^{2t} \sin(3t)$

(b) (4 points). Now consider the inhomogeneous equation

$$y'' + 4y' + 29y = 13e^{-3t}.$$

Which one of the following functions is a particular solution of that equation?

☐ $t^2 + e^{-3t+1}$	☐ $2e^{3t}$	☐ $\frac{1}{3}e^{3t}$	☐ $\frac{1}{2}e^{-3t}$
☐ $2e^{-3t}$	☐ $\frac{1}{4}e^{-3t}$	☐ $\frac{1}{9}e^{t} + e^{-t}$	☐ $\frac{2}{3}e^{-3t+1}$

Problem 2 (6 points)

A surface $\mathcal{F}$ in space is determined by the equation $F(x, y, z) = 0$ where

$$F(x, y, z) = x^3 - y^3 + 2z^3 - 9.$$

The surface $\mathcal{F}$ has a tangent plane at the point $P = (2, 1, 1)$. Mark an equation for this tangent plane among the following options:

- ☐ $3x - 3y + 12z = 15$
- ☐ $2x - 2y + 7z = 5$
- ☐ $x - 2y + z = 1$
- ☐ $-3x + 2y + 6z = 11$
- ☐ $9x + y + 6z = 25$
- ☐ $x - 2y - z = 1$
- ☐ $12x - 3y + 6z = 27$
- ☐ $x - 3y + 5z = -3$

Problem 3 (10 points)

A plane curve is given by

$$x = t - 2t^2,$$
$$y = 2t + t^2,$$

where the parameter t runs through the real numbers.

(a) (2 points). Which point on the curve corresponds to the parameter value $t = 1$?

- ☐ $(1, 4)$
- ☐ $(-1, 1)$
- ☐ $(-1, 2)$
- ☐ $(3, 4)$
- ☐ $(-1, 3)$
- ☐ $(5, 4)$
- ☐ $(-3, 4)$
- ☐ $(-5, 4)$

(b) (4 points). What is the curvature of the curve for $t = 1$?

- ☐ $\frac{\sqrt{2}}{4}$
- ☐ $\frac{\sqrt{3}}{4}$
- ☐ $\frac{2}{25}$
- ☐ 5
- ☐ $\frac{1}{2}$
- ☐ $\sqrt{3}$
- ☐ $\frac{1}{9}$
- ☐ $\sqrt{11}$

(c) (4 points). For which value of t is the curvature maximal?

- ☐ -2
- ☐ 0
- ☐ 3
- ☐ 5
- ☐ $-\frac{1}{2}$
- ☐ $\frac{3}{2}$
- ☐ $\frac{7}{2}$
- ☐ 9

Problem 4 (7 points)

A function is defined by

$$f(x) = \left(4x+1\right)^{\frac{1}{2}}.$$

(a) (3 points). What is the double derivative $f''(x)$?

- ☐ $\left(4x+1\right)^{-\frac{1}{2}}$
- ☐ $2x^{-\frac{1}{2}}$
- ☐ $\frac{1}{4}\left(4x+1\right)^{\frac{1}{2}}$
- ☐ $-\frac{1}{4}\left(4x+1\right)^{-\frac{3}{2}}$
- ☐ $6x^{-\frac{3}{2}}$
- ☐ $-4\left(4x+1\right)^{-\frac{3}{2}}$

(b) (4 points). Which one of the polynomials below is the third order Taylor polynomial for f at the point $x=0$?

- ☐ $1+2x-\frac{1}{2}x^2+\frac{1}{6}x^3$
- ☐ $1-\frac{1}{2}x^3$
- ☐ $2+4x^2-\frac{1}{2}x^3$
- ☐ $1-x+x^2-6x^3$
- ☐ $1+3x+x^2+2x^3$
- ☐ $1+2x-2x^2+4x^3$

Problem 5 (10 points)

A particle is moving along a curve in space. The position vector of the particle at time t is

$$\mathbf{r}(t) = \langle e^t,\, e^{-t},\, \sqrt{2}\,t\rangle.$$

(a) (3 points). What is the velocity vector $\mathbf{v}(t)$?

- ☐ $\langle e^t,\, -e^{-t},\, \frac{1}{2\sqrt{2}}t\rangle$
- ☐ $\langle te^{t-1},\, -te^{-t-1},\, 0\rangle$
- ☐ $\langle e^t,\, e^{-t}, 1\,\rangle$
- ☐ $\langle e^t,\, -e^{-t},\, \sqrt{2}\rangle$
- ☐ $\langle e^t,\, -e^{-t}, \frac{\sqrt{2}}{2}t^2\,\rangle$
- ☐ $\langle e^t,\, e^{-t},\, t\rangle$

(b) (4 points). What is the speed $v(t)$?

- ☐ e^t+e^{-t}
- ☐ $\sqrt{e^{2t}+e^{-2t}+\frac{1}{8}t}$
- ☐ $e^t+e^{-t}+\sqrt{2}$
- ☐ $2e^{2t}+2t$
- ☐ $\sqrt{e^{2t}+e^{-2t}+t^2}$
- ☐ $\sqrt{e^{2t}+e^{-2t}+2t^2}$

(c) (3 points). The particle runs through a segment of the motion curve during the time interval $0 \le t \le 1$. What is the arc length of that segment?

- ☐ e^2
- ☐ $e-e^{-1}$
- ☐ $2e+1$
- ☐ $e-e^{-1}+\sqrt{2}$
- ☐ $\sqrt{2}$
- ☐ $3\ln(2)$

Problem 6 (6 points)

Two complex numbers are given by

$$z_1 = \frac{1+i}{3-2i} \quad , \quad z_2 = \frac{2i}{1+i}$$

(a) (3 points). What is z_1 written in standard form?

☐ $2+3i$	☐ $\frac{1}{13}+\frac{5}{13}i$	☐ $8i$	☐ $\frac{1}{4}-\frac{3}{4}i$
☐ $\frac{1}{5}-\frac{2}{5}i$	☐ $\frac{2}{5}+\frac{3}{5}i$	☐ -5	☐ $-2-i$

(b) (3 points). What is z_2 written in polar form?

☐ $2e^{i\pi/3}$	☐ $2e^{i\pi}$	☐ $2e^{i\pi/2}$	☐ $\sqrt{6}e^{i\pi/4}$
☐ $\sqrt{2}e^{i\pi/4}$	☐ $\sqrt{2}e^{-i\pi/3}$	☐ $\frac{1}{2}e^{-i\pi}$	☐ $5e^{3i\pi/4}$

Problem 7 (8 points)

A function is defined as

$$f(x,y) = \frac{4x-2y^2-3}{x^2-y^2}.$$

Mark the correct option in each question below.

(a) (4 points). The domain of f consists of all those points (x,y) which satisfy

☐ $x>0$ and $y>0$	☐ $y \neq x$ and $y \neq -x$
☐ $x>y$	☐ $x \neq 0$ and $y \neq 0$
☐ $4x+5 \geq 0$	☐ $x \neq 0$ or $y \neq 0$

(b) (4 points). The level curve with equation $f(x,y)=1$ can be described as:

☐ A parabola with equation $y = 2x^2 + \frac{5}{2}$.

☐ A parabola with equation $y = x^2 - 2x + 4$.

☐ A straight line through $(0,5)$ with slope 4.

☐ A straight line through $(0,2)$ with slope 2.

☐ A circle with center at $(-2,1)$ and radius $\frac{1}{2}$.

☐ A circle with center at $(2,0)$ and radius 1.

Problem 8 (14 points)

A function is defined as

$$f(x,y) = xy^2 - 6x^2 - 3y^2.$$

(a) (2 points). What is the function value $f(1,1)$?

☐ -10	☐ 0	☐ 3	☐ 7
☐ -8	☐ 2	☐ 5	☐ 11

(b) (4 points). Which of the following points are critical points for f? (*Remark*: Each wrong mark in this question will cancel a true mark.)

☐ $(0,0)$	☐ $(-1,1)$	☐ $(2,3)$	☐ $(4,2)$
☐ $(1,1)$	☐ $(3,6)$	☐ $(3,-6)$	☐ $(4,-16)$

(c) (4 points). What is the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (-1,1)$ and in the direction of the unit vector $\mathbf{u} = \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j}) = \frac{\sqrt{2}}{2}\langle 1,1\rangle$?

☐ $2\sqrt{2}$	☐ $\frac{\sqrt{2}}{4}$	☐ 8	☐ $\frac{\sqrt{6}}{2}$
☐ $\frac{5\sqrt{2}}{2}$	☐ $10\sqrt{2}$	☐ -7	☐ $\frac{1}{3}$

(d) (4 points). Which one of the following equations is an equation for the tangent plane to the graph of f at the point $Q = (1,1,f(1,1))$?

☐ $x + 2y - z = 11$	☐ $11x + 4y + z = 7$
☐ $2x - 3y + z = -9$	☐ $x + y - z = 1$
☐ $9x + 10y - z = 13$	☐ $10x + y + z = 3$

Problem 9 (5 points)

Let

$$g(x,y,z) = \arctan(x^2 + y^2 - z^2).$$

In which direction is the directional derivative of g at $P = (1,-1,1)$ maximal? Mark a unit vector below which specifies the direction.

☐ $\frac{\sqrt{3}}{3}\langle 1,-1,-1\rangle$	☐ $\frac{1}{3}\langle 2,2,1\rangle$
☐ $\frac{\sqrt{2}}{2}\langle 1,1,0\rangle$	☐ $\frac{1}{3}\langle -2,1,2\rangle$
☐ $\frac{\sqrt{2}}{2}\langle -1,0,1\rangle$	☐ $\frac{1}{5}\langle 3,-4,0\rangle$

Problem 10 (10 points)

A plane region $\mathcal{R}$ consists of those points (x, y) which satisfy the inequalities

$$0 \leq x, \quad 0 \leq y, \quad x^2 + y^2 \leq 16.$$

Mark the value of the double integral

$$\iint_{\mathcal{R}} x\sqrt{x^2 + y^2}\, dA.$$

☐ $\frac{5}{4}$ ☐ $\frac{55}{4}$ ☐ 8 ☐ 64 ☐ 91

☐ $\frac{5}{2}$ ☐ $\frac{47}{2}$ ☐ 55 ☐ 72 ☐ 121

Problem 11 (10 points)

A space region $\mathcal{T}$ consists of those points (x, y, z) which satisfy the inequalities

$$0 \leq x \leq 1, \quad 0 \leq y \leq 2, \quad 0 \leq z \leq 5 - xy.$$

A solid body with density function $\delta(x, y, z) = x$ covers $\mathcal{T}$ precisely.

(a) (5 points). What is the volume of $\mathcal{T}$?

☐ $\frac{1}{3}$ ☐ $\frac{3}{2}$ ☐ $\frac{22}{3}$ ☐ 3 ☐ 9

☐ $\frac{1}{2}$ ☐ $\frac{7}{2}$ ☐ 2 ☐ 5 ☐ 11

(b) (5 points). What is the mass of the solid body covering $\mathcal{T}$?

☐ $\frac{13}{3}$ ☐ $\frac{15}{2}$ ☐ 3 ☐ 5 ☐ 10π

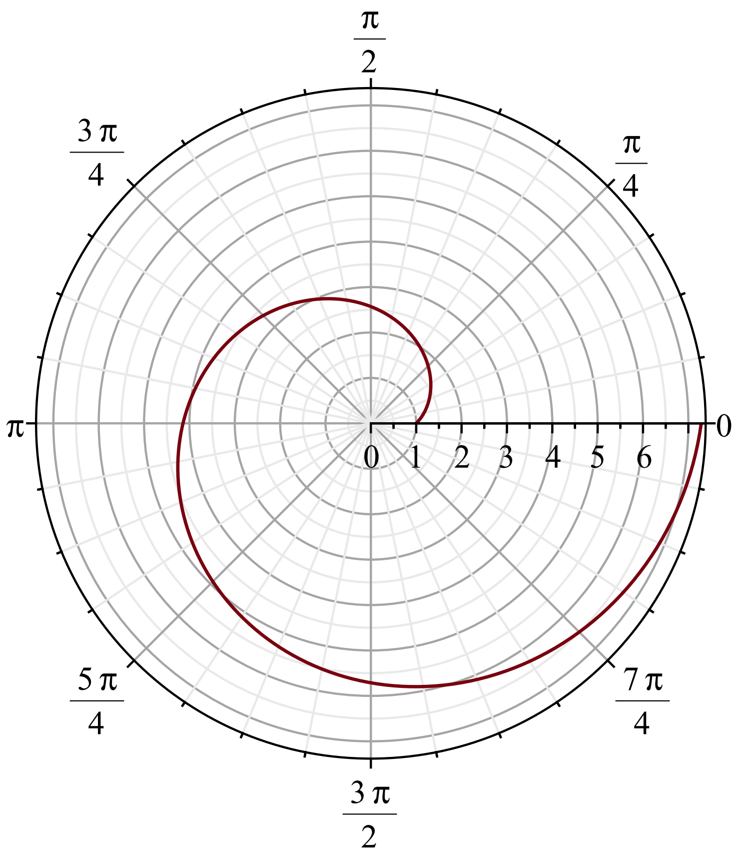
☐ $\frac{11}{2}$ ☐ 2 ☐ 4 ☐ 8 ☐ $\frac{4\pi}{7}$

Problem 12 (5 points)

The figure below shows the graph of the function

$$r = f(\theta), \quad 0 \le \theta \le 2\pi$$

in polar coordinates



Which one of the following rules for f corresponds to the figure?

- ☐ $f(\theta) = 1 + \sin\theta$
- ☐ $f(\theta) = \sin(2\theta)$
- ☐ $f(\theta) = 1 + \theta$
- ☐ $f(\theta) = 1 + \cos\theta$
- ☐ $f(\theta) = 1 - 2\sin\theta$
- ☐ $f(\theta) = 1 + \theta^2$