Exam in Calculus

First Year at The Faculty of Engineering and Science
and The Faculty of Medicine

6 June 2016

The present exam set consists of 10 numbered pages with 14 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use books, notes etc. It **is not** allowed to use electronic devices.

Your answers must be marked directly in this exam set. The grading of the exam is based solely on these marks. Remember to write your **full name** and **student number** below.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Problem 1 (7 points)

Consider the differential equation

$$y'' + 2y' + 26y = 0.$$

A number of function expressions, which contain two arbitrary constants c_1 and c_2, are listed below. Mark the expression which constitute the complete solution of the differential equation.

☐ $y(t) = c_1 e^{2t} + c_2 e^{-2t}$

☐ $y(t) = c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t)$

☐ $y(t) = c_1 e^{t} + c_2 e^{3t}$

☐ $y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

☐ $y(t) = c_1 e^{-t}\cos(5t) + c_2 e^{-t}\sin(5t)$

☐ $y(t) = c_1 e^{2t} + c_2 t e^{2t}$

☐ $y(t) = c_1 e^{2t}\cos(26t) + c_2 e^{2t}\sin(26t)$

☐ $y(t) = c_1 e^{-4t} + c_2 e^{-2t}$

☐ $y(t) = c_1 e^{t}\cos(3t) + c_2 e^{t}\sin(3t)$

☐ $y(t) = c_1 e^{-t}\cos(2\pi t) + c_2 e^{-t}\sin(2\pi t)$

Problem 2 (7 points)

The complete solution of the differential equation

$$y'' - 2y' + y = 0$$

can be written as

$$y(t) = c_1 e^t + c_2 t e^t$$

where c_1 and c_2 are arbitrary constants.

(a) (1 point). Mark the correct expression for $y(0)$ below

☐ $c_1 + c_2$ ☐ $c_1 - c_2$ ☐ c_1 ☐ $2c_1$ ☐ 0

(b) (3 points). Mark the correct expression for $y'(0)$ below

☐ c_1 ☐ $c_1 + c_2$ ☐ c_2 ☐ $c_1 + 2c_2$ ☐ $c_1 - 2c_2$

(c) (3 points). The initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 4$$

has a unique solution $y(t)$. Find this solution and indicate the function value $y(1)$ below.

☐ 5 ☐ $5e$ ☐ $-3e$ ☐ 4 ☐ $4e$

Problem 3 (7 points)

A function is defined by

$$f(x) = x\cos(x).$$

(a) (3 points). Mark the correct expression for the double derivative $f''(x)$.

- ☐ $-2\sin(x) - x\cos(x)$
- ☐ $\cos(x) - \sin(x)$
- ☐ $\sin(x) - x\cos(x)$
- ☐ $-\sin(x)$
- ☐ 0
- ☐ $-\cos(x)$

(b) (4 points). Which of the polynomials below is the 2nd order Taylor polynomial for $f(x)$ about the point $a = 0$?

- ☐ $x - \frac{1}{2}x^2$
- ☐ $-x + x^2$
- ☐ x^2
- ☐ $1 + x^2$
- ☐ x
- ☐ $1 - 3x - 4x^2$
- ☐ $2x + 5x^2$
- ☐ $3x$
- ☐ $x + 9x^2$
- ☐ $1 + x + x^2$

Problem 4 (6 points)

A curve in space is given by

$$x = \cos(t),$$
$$y = 2\sin(t),$$
$$z = t,$$

where the parameter t runs through the real numbers. Mark the correct expression for the arc length of the curve from $t = 0$ to $t = \pi$.

- ☐ $\int_0^{\pi} \sqrt{2}\,dt$
- ☐ $\int_0^{\pi} \sqrt{\cos^2(t) + 4\sin^2(t) + t^2}\,dt$
- ☐ $\int_0^{\pi} (-\sin(t) + 2\cos(t) + 1)\,dt$
- ☐ $\int_0^{\pi} (\sin(t) + 2\cos(t) + 1)\,dt$
- ☐ $\int_0^{\pi} \sqrt{3\cos^2(t) + 2}\,dt$
- ☐ $\int_0^{\pi} \sqrt{3 + t^2}\,dt$

Problem 5 (8 points)

A plane curve is given by

$$x = \sin(t),$$
$$y = t^2,$$

where the parameter t runs through the real numbers.

(a) (1 point). Which point on the curve corresponds to the parameter value $t = 0$?

☐ $(\pi, 0)$ ☐ $(1, 1)$ ☐ $(-1, 1)$ ☐ $(0, 0)$ ☐ $(1, 0)$

(b) (7 points). What is the curvature of the curve for $t = 0$?

☐ $\frac{3}{125}$ ☐ 3 ☐ 2 ☐ $\frac{1}{2}$ ☐ $\frac{\sqrt{2}}{2}$

Problem 6 (10 points)

A function is given by

$$f(x, y) = \frac{x^2 + y^2 + 1}{2x + y}.$$

Mark the correct option in each of the subquestions below.

(a) (5 points). The domain for f consists of all points (x, y) which satisfy

☐ $x \neq 0$ and $y \neq 0$ ☐ $y \neq -2x$

☐ $x^2 + y^2 \leq 1$ ☐ $x \geq 0$ and $y \geq 0$

☐ $x^2 + y^2 \leq 2$ ☐ $2x + y = 1$

(b) (5 points). The level curve with the equation $f(x, y) = 2$ can be described as:

☐ A circle with center at $(1, 2)$ and radius 1.

☐ A circle with center at $(2, 1)$ and radius 2.

☐ A straight line through $(0, 0)$ with slope 2.

☐ A straight line through $(0, 0)$ with slope $\frac{1}{2}$.

☐ A parabola with equation $y = 3x^2 + 3x + 1$.

☐ A parabola with equation $y = 5x^2 + 1$.

Problem 7 (6 points)

A funktion is defined by

$$f(x,y) = \sin(x^3 + x^2y + y^2 - 1).$$

(a) (2 points). Indicate the function value $f(1,-1)$ below.

☐ -1 ☐ 1 ☐ π ☐ $\frac{\pi}{2}$ ☐ 0

(b) (4 point). Mark the correct expression for the partial derivative $f_x(x,y)$.

☐ $3x^2\cos(x^3 + x^2y + y^2 - 1)$

☐ $\sin(3x^2 + 2xy)$

☐ $\sin(3x^2)$

☐ $\cos(3x^2 + 2xy)$

☐ $(3x^2 + 2xy)\cos(x^3 + x^2y + y^2 - 1)$

☐ $\cos(x^3 + x^2y + y^2 - 1)$

Problem 8 (8 points)

A function is given by

$$f(x,y) = x^2 - 2y^2 + 3xy - 8x + 5y + 1.$$

(a) (4 points). Which one of the following points is a critical point for f?

☐ $(0,0)$ ☐ $(1,0)$ ☐ $(0,2)$ ☐ $(1,2)$ ☐ $(1,1)$

(b) (4 points). What is the value of the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (1,1)$ and in the direction of the unit vector $\mathbf{u} = -\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j} = (-\frac{4}{5}, \frac{3}{5})$?

☐ $\frac{3}{5}$ ☐ 0 ☐ $\frac{24}{5}$ ☐ 3 ☐ $-\frac{11}{5}$

Problem 9 (7 points)

A surface in space $\mathcal{F}$ is given by the equation $F(x,y,z) = 0$ where

$$F(x,y,z) = e^x + y^2 + z^3 - 6.$$

The surface $\mathcal{F}$ has a tangent plane at the point $P = (0,2,1)$. Mark an equation of this tangent plane below.

- ☐ $x + y + 3z = 4$
- ☐ $x + 4y + 3z = 11$
- ☐ $2x - y + 3z = 1$
- ☐ $3x + 2y + 3z = 3$
- ☐ $x + y - z = 5$
- ☐ $2x + y - 6z = 10$
- ☐ $x - 2y + 6z = 12$
- ☐ $6x + y + z = 3$
- ☐ $x + y + z = 5$
- ☐ $x + y - z = 11$

Problem 10 (10 points)

A region $\mathcal{R}$ in the plane consists of those points (x,y) which satisfy the inequalities

$$x^2 + y^2 \leq 1, \quad 0 \leq y.$$

Mark the value of the double integral

$$\iint_{\mathcal{R}} (x^2 + y^2)^3 \, dA.$$

- ☐ $\frac{2\pi}{3}$
- ☐ 6
- ☐ $\frac{\sqrt{3}}{2}$
- ☐ $\frac{6\pi}{5}$
- ☐ $\frac{\pi}{4}$
- ☐ $\frac{\pi}{12}$
- ☐ $\frac{3}{4}$
- ☐ $\frac{\pi}{8}$
- ☐ 9
- ☐ $\frac{1}{2}$

Problem 11 (5 points)

Two complex numbers are given by

$$z_1 = \frac{3+i}{1+2i} + 7 - i, \qquad z_2 = 5e^{3\pi i}.$$

(a) (3 points). What is z_1 written in standard form?

☐ $8+i$ ☐ $3i$ ☐ $5+2i$ ☐ $8-2i$ ☐ $7+i$

(b) (2 points). What is z_2 written in standard form?

☐ $5+5i$ ☐ -5 ☐ $5-5i$ ☐ $5i$ ☐ 5

Remark. In problem 12 the evaluation of your answers will be performed using the following principle: Every false mark cancels a true mark.

Problem 12 (6 points)

Let $\mathcal{T}$ be the region in space consisting of those points (x, y, z) which satisfy the inequalities

$$0 \leq x \leq 1, \quad 0 \leq y \leq 2-2x, \quad 0 \leq z \leq 3x.$$

A solid body with density function $\delta(x, y, z) = x + 1$ covers region $\mathcal{T}$ precisely. The volume of the body is denoted V and its mass is denoted m. Mark all of the correct expressions below.

☐ $m = \int_0^{2-2x} \int_0^1 \int_0^{3x} (x+1)\, dzdxdy.$

☐ $m = \int_0^1 \int_0^{2-2x} \int_0^{2+x} (x+1)\, dzdydx.$

☐ $m = \int_0^1 \int_0^{2-2x} \int_0^{3x} (x+1)\, dzdydx.$

☐ $V = \int_0^2 \int_0^{\frac{2-y}{2}} \int_0^{3x} dzdxdy.$

☐ $V = \int_0^2 \int_0^{1-y} \int_0^{3x} dzdxdy.$

Problem 13 (8 points).

Answer the following 4 True/False problems:

(a) (2 points). The point with rectangular coordinates $(x, y) = (2, -2)$ can be described by $(r, \theta) = (2\sqrt{2}, \frac{\pi}{2})$ in polar coordinates.

☐ True ☐ False

(b) (2 points). Let$D$ be the region in the plane consisting of those points (x, y) which satisfy the inequalities $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Let f be the function with rule

$$f(x, y) = \frac{2x}{x + y + 1}$$

and domain D. Then f attains a global maximum on D.

☐ True ☐ False

(c) (2 points).Let$D$ be the region in the plane consisting of those points (x, y) which satisfy the inequalities $0 < x \leq 1$ and $0 \leq y \leq 1$. Let f be the function with rule

$$f(x, y) = \frac{1}{x + y}$$

and domain D. Then f attains a global maximum on D.

☐ True ☐ False

(d) (2 points). For every real number b the following equation in the unknown x has precisely one solution:

$$\arctan(x) = b.$$

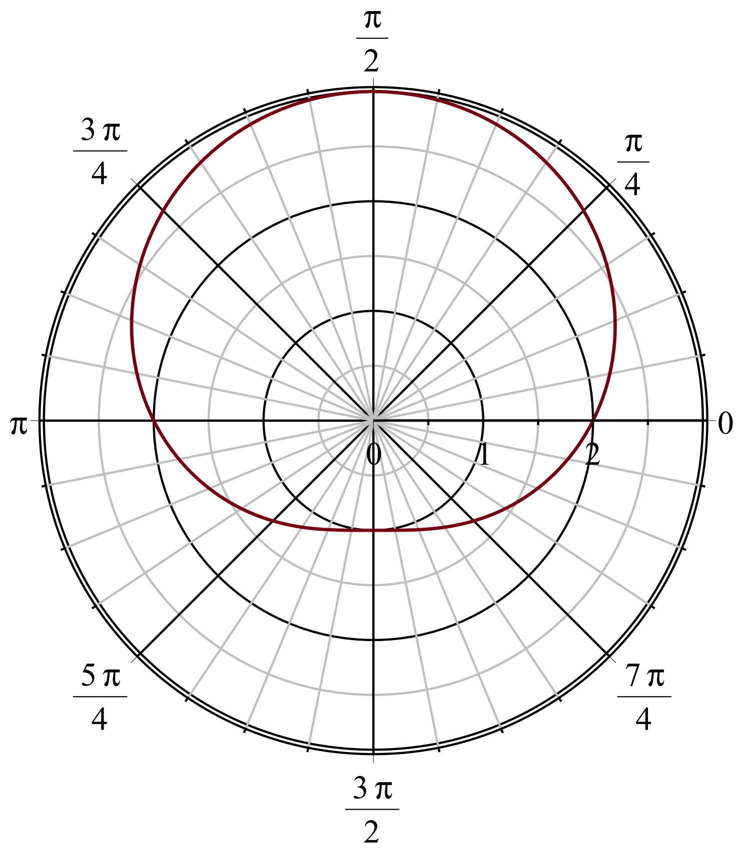
☐ True ☐ False

Problem 14 (5 points)

The figure below shows the graph of the function

$$r = f(\theta), \quad 0 \le \theta \le 2\pi$$

in polar coordinates.



Which one of the following rules for f corresponds to this figure?

- ☐ $f(\theta) = 3 - \cos(\theta)$
- ☐ $f(\theta) = 2 - \cos(\theta)$
- ☐ $f(\theta) = 2 + \sin(\theta)$
- ☐ $f(\theta) = 1 + \sin(\theta)$
- ☐ $f(\theta) = 2\sin(\theta)$
- ☐ $f(\theta) = 3\cos(2\theta)$