Exam in Calculus

First Year at the Faculty of Engineering and Science and the Technical Faculty of IT and Design

3 January 2017

The present exam set consists of 9 numbered pages with 13 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use books, notes etc. It **is not** allowed to use electronic devices.

Your answers must be marked directly in this exam set. The grading of the exam is based solely on these marks. Remember to write your **full name** and **student number** below.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Problem 1 (9 points)

A plane curve is given by

$$x = t^2,$$
$$y = t^3 - t,$$

where the parameter t runs through the real numbers.

(a) (2 points). The curve has a self-intersection at a point with first coordinate $x = 1$. What is the second coordinate of this intersection point?

☐ 1 ☐ 3 ☐ -1 ☐ 0 ☐ 2

(b) (7 points). What is the curvature of the curve for $t = 0$?

☐ 3 ☐ $\frac{1}{2}$ ☐ π ☐ $\frac{\sqrt{3}}{2}$ ☐ 0

☐ $\frac{\sqrt{2}}{2}$ ☐ 1 ☐ 2 ☐ 4 ☐ $\sqrt{3}$

Problem 2 (9 points)

A curve in space is given by

$$x = \tfrac{1}{2}t^2,$$
$$y = \tfrac{1}{3}(2t)^{\frac{3}{2}},$$
$$z = t,$$

where the parameter t runs through the positive real numbers.

(a) (2 points). Mark the correct expression for the derivative y'.

☐ $t\sqrt{2t}$ ☐ t ☐ $\sqrt{2t}$ ☐ $t\sqrt{t}$ ☐ $\sqrt{t}$

(b) (7 points). What is the arc length of the curve from $t = 2$ to $t = 4$?

☐ 8 ☐ $\frac{\sqrt{2}}{3}$ ☐ 10 ☐ $\sqrt{3}$ ☐ 1

☐ $\sqrt{2}$ ☐ 6 ☐ $\frac{5}{2}$ ☐ $\frac{8}{3}$ ☐ 12

Problem 3 (7 points)

A function is defined by

$$f(x) = \ln(\cos x)$$

for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(a) (2 points). Mark the correct expression for the derivative $f'(x)$.

- ☐ $\frac{1}{\cos x}$
- ☐ $-\frac{1}{\sin x}$
- ☐ $\frac{1}{x}$
- ☐ $\frac{\sin x}{x}$
- ☐ $-\tan x$
- ☐ $\frac{\cos x}{x}$

(b) (5 points). Which one of the polynomials below is the second order Taylor polynomial for $f(x)$ about the point $x = 0$?

- ☐ $x + \frac{3}{2}x^2$
- ☐ $1 - 3x$
- ☐ $-\frac{1}{2}x^2$
- ☐ $1 + 3x^2$
- ☐ $1 + x - \frac{3}{2}x^2$
- ☐ $x - x^2$
- ☐ $2x + \frac{5}{2}x^2$
- ☐ $2 + x - 3x^2$
- ☐ x^2
- ☐ $1 - x + x^2$

Problem 4 (5 points)

Two complex numbers are given by

$$z_1 = (1 - i)(2 - 3i) + 7i, \quad z_2 = i(1 + i).$$

(a) (2 points). What is z_1 written in standard form?

☐ $2 + 10i$ ☐ $1 + i$ ☐ $-1 + 2i$ ☐ i ☐ $3 - i$

(b) (3 points). What is z_2 written in polar form?

☐ $\sqrt{2}\, e^{3\pi i/4}$ ☐ $\sqrt{2}\, e^{\pi i/2}$ ☐ $2e^{\pi i/4}$ ☐ $e^{-\pi i/4}$ ☐ e^{2i}

Problem 5 (10 points)

(a) (6 points). A homogeneous second order differential equation is given by

$$y'' + 4y' + 20y = 0.$$

A number of function expressions, which contain two arbitrary constants c_1 and c_2, are listed below. Mark the expression which constitute the general solution of the differential equation.

☐ $y(t) = c_1 e^{4t} + c_2 e^{5t}$

☐ $y(t) = c_1 e^{2t} + c_2 t e^{2t}$

☐ $y(t) = c_1 e^{-2t} + c_2 e^{3t}$

☐ $y(t) = c_1 e^{3t} + c_2 t e^{3t}$

☐ $y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

☐ $y(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)$

☐ $y(t) = c_1 e^{-t}\cos(5t) + c_2 e^{-t}\sin(5t)$

☐ $y(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t)$

(b) (4 points). The function $f(t) = \frac{1}{4}t$ is seen to be a particular solution to the inhomogeneous differential equation

$$y'' + 4y' + 20y = 1 + 5t.$$

Mark a particular solution to the differential equation

$$y'' + 4y' + 20y = 1 + 5t + 5e^t$$

among the following function expressions:

☐ $g(t) = 2t + e^t$

☐ $g(t) = \frac{1}{4}t + \frac{1}{5}e^t$

☐ $g(t) = \frac{1}{4}t + e^{5t}$

☐ $g(t) = t^2 + e^{-t}$

☐ $g(t) = \frac{1}{4}t + 3e^t$

☐ $g(t) = \frac{1}{4}t + \frac{1}{2}e^{10t}$

☐ $g(t) = \frac{1}{8}t + \frac{1}{8}t^2 + e^t$

☐ $g(t) = \frac{1}{4}t + 5e^t$

Problem 6 (9 points)

A function is given by

$$f(x,y) = \frac{2x^2 - 2y + 3}{x^2 - y}.$$

Mark the correct option in each of the subquestions below.

(a) (4 points). The domain for f consists of all points (x,y) which satisfy

- ☐ $x^2 > y$
- ☐ $y \leq x$
- ☐ $y \neq x^2$
- ☐ $2x - 1 \neq 0$
- ☐ $x^2 + y^2 \leq 1$
- ☐ $y \leq 0$

(b) (5 points). The level curve with equation $f(x,y) = 5$ can be described as:

- ☐ A parabola with equation $y = x^2 - 1$.
- ☐ A parabola with equation $y = 2x^2 - y + 3$.
- ☐ A straight line through $(0,0)$ with slope 3.
- ☐ A straight line through $(0,3)$ with slope 2.
- ☐ A circle with center $(0, \frac{3}{2})$ and radius 2.
- ☐ A circle with center $(1,2)$ and radius $\frac{1}{2}$.

Problem 7 (5 points)

A function is defined by

$$f(x,y) = x^2 \arctan y.$$

Mark the correct expression for the second order partial derivative $f_{xy}(x,y)$.

- ☐ $2x(1 + \tan^2 y)$
- ☐ $\frac{2x}{\sqrt{1-y^2}}$
- ☐ $\frac{2}{1+y^2}$
- ☐ $2 \arctan y$
- ☐ $\frac{2x}{1+y^2}$
- ☐ $\frac{-2x^2 y}{(1+y^2)^2}$

Problem 8 (12 points)

A function is defined by

$$f(x,y) = x^3 + y - xy.$$

(a) (1 point). Indicate the function value $f(-1,1)$.

☐ 1 ☐ -1 ☐ 0 ☐ 3 ☐ -3

(b) (3 points). Which one of the following points is a critical point for f?

☐ $(1,1)$ ☐ $(-1,2)$ ☐ $(1,3)$ ☐ $(2,1)$ ☐ $(1,2)$

(c) (4 points). The graph of f has a tangent plane at the point $P = (-1,1,f(-1,1))$. Mark an equation for this tangent plane.

☐ $2x + 2y - z = -1$ ☐ $x + 2y - z = 3$

☐ $x + 2y + z = 2$ ☐ $x - z = 3$

☐ $-x + y - z = 1$ ☐ $3x + 2y - z = 2$

(d) (4 points). Let $g(t)$ and $h(t)$ be two differentiable functions satisfying the conditions $g(0) = 2$, $g'(0) = 1$ and $h(0) = 0$, $h'(0) = 2$. Consider the composite function

$$w(t) = f(g(t), h(t)).$$

What is the value of the derivative $w'(0)$?

☐ 0 ☐ 13 ☐ 2 ☐ -1 ☐ $\frac{1}{2}$

☐ 7 ☐ 10 ☐ 3 ☐ 8 ☐ $-\frac{3}{2}$

Problem 9 (5 points)

A function is defined by

$$f(x,y,z) = x^2 - e^y + e^{z+1}.$$

What is the value of the directional derivative $D_{\mathbf{u}}f(P)$ at the point $P = (1,0,-1)$ and in the direction of the unit vector $\mathbf{u} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$?

☐ $\frac{1}{3}$ ☐ $\frac{4}{3}$ ☐ -3 ☐ $-\frac{\sqrt{2}}{2}$ ☐ $\frac{1}{2}$

☐ 2 ☐ $\frac{\sqrt{3}}{3}$ ☐ 1 ☐ $\frac{2}{3}$ ☐ 7

Problem 10 (10 points)

A region $\mathcal{R}$ in the plane consists of those points (x, y) which satisfy the inequalities

$$x^2 + y^2 \leq \frac{\pi}{2}, \quad 0 \leq y.$$

Mark the value of the double integral

$$\iint_{\mathcal{R}} \cos(x^2 + y^2)\, dA.$$

☐ $\frac{1}{2}$	☐ $\frac{4\pi}{3}$	☐ $\frac{11}{7}$	☐ $\frac{\pi}{2}$	☐ $\frac{3\pi}{11}$
☐ $\frac{\pi^2}{4}$	☐ $\frac{2\pi}{7}$	☐ $\frac{\pi}{4}$	☐ $\frac{5}{2}$	☐ 1

Remark. In problem 11 the evaluation of your answers will be performed using the following principle: Each false mark cancels a true mark.

Problem 11 (6 points)

Let $\mathcal{T}$ be the region in space consisting of those points (x, y, z) which satisfy the inequalities

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x, \quad 0 \leq z \leq 1 - x - y.$$

A solid body with density function $\delta(x, y, z) = 1 - z$ covers region $\mathcal{T}$ precisely. The volume of the body is denoted V and its mass is denoted m. Mark all of the correct expressions below.

☐ $m = \int_0^{1-x-y} \int_0^{1-x} \int_0^{1} (1 - z)\, dxdydz.$

☐ $m = \int_0^{1} \int_0^{1} \int_0^{1-x-y} (1 - z)\, dzdydx.$

☐ $m = \int_0^{1} \int_0^{1-x} \int_0^{1-x-y} (1 - z)\, dzdydx.$

☐ $V = \int_0^{1} \int_0^{1-y} \int_0^{1-x-y} dzdxdy.$

☐ $V = \int_0^{1-x} \int_0^{1} \int_0^{1-x-y} dzdxdy.$

Problem 12 (8 points).

Answer the following 4 True/False problems:

(a) (2 points). The equation

$$\frac{d}{dx}\big((x^2+1)\arctan x\big) = 2x\arctan x + 1$$

holds for every real number x.

☐ True ☐ False

(b) (2 points). The point with polar coordinates $(r,\theta) = (-5,\pi)$ has rectangular coordinates $(x,y) = (-5,0)$.

☐ True ☐ False

(c) (2 points). For every complex number z one has

$$|iz| = -|z|.$$

☐ True ☐ False

(d) (2 points). Let D be the region in the plane consisting of those points (x,y) which satisfy the inequality $x^2+y^2 \leq 4$. Let f be the function with rule

$$f(x,y) = x^2y - e^x$$

and domain D. Then f attains a global minimum on D.

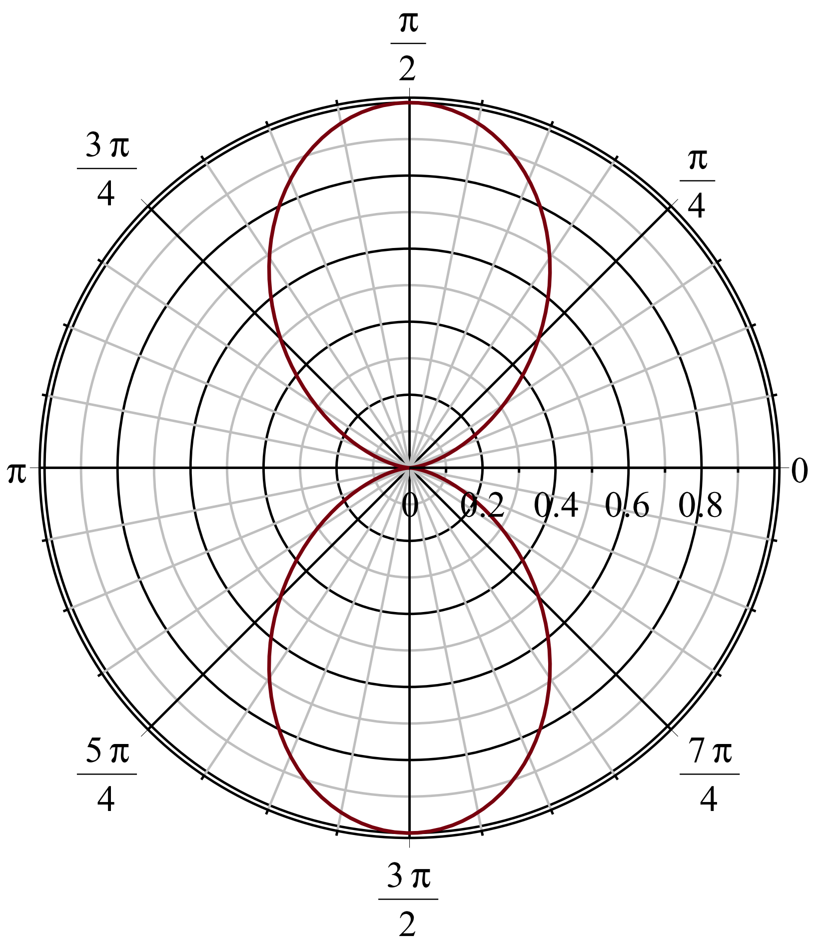
☐ True ☐ False

Opgave 13 (5 points)

The figure below shows the graph of the function

$$r = f(\theta), \quad 0 \leq \theta \leq 2\pi$$

in polar coordinates.



Which one of the following rules for f corresponds to the figure?

- ☐ $f(\theta) = \sin(2\theta)$
- ☐ $f(\theta) = 1 - \cos\theta$
- ☐ $f(\theta) = \sin^2\theta$
- ☐ $f(\theta) = 2 - \sin\theta$
- ☐ $f(\theta) = 1 + 2\sin\theta$
- ☐ $f(\theta) = \cos^2\theta$