Om formler for cirkulære funktioner

I Kompendium i calculus er der på side 4 angivet en række formler for cirkulære funktioner. Her følger nogle overvejelser om fremkomst og udvælgelse af formlerne.

Først en oversigt over diverse formler for cirkulære funktioner med angivelse af gyldighedsinterval:

$$\arcsin x = \arctan \frac{x}{\sqrt{1-x^2}}, \qquad -1 < x < 1 \tag{1}$$

$$\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}}, \qquad -\infty < x < \infty \tag{2}$$

$$\arccos x = \arctan \frac{\sqrt{1-x^2}}{x}, \qquad 0 < x \leq 1 \tag{3}$$

$$\arctan x = \arccos \frac{1}{\sqrt{1+x^2}}, \qquad 0 \leq x < \infty \tag{4}$$

$$\arctan x = \frac{1}{2} \arcsin \frac{2x}{1+x^2}, \qquad -1 \leq x \leq 1 \tag{5}$$

$$\arctan x = \frac{1}{2} \arccos \frac{1-x^2}{1+x^2}, \qquad 0 \leq x < \infty \tag{6}$$

Kommentarer til formlernes anvendelighed:

- (1) og (2) har fuldt gyldighedsinterval.
- (3) kan erstattes af

$$\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - \arctan \frac{x}{\sqrt{1-x^2}}, \quad -1 < x < 1,$$

som har fuldt gyldighedsinterval.

- (4) og (6) omhandler begge arctan udtrykt ved arccos. Den simpleste form med fuldt gyldighedsinterval fremkommer ved at “reparere” på (6):

$$\arctan x = \pm\frac{1}{2} \arccos \frac{1-x^2}{1+x^2}, \qquad \begin{cases} + & \text{for } 0 \leq x < \infty \\ - & \text{for } -\infty < x \leq 0 \end{cases}$$

- (5) er dækket ind af (2) og sådan set “overflødig” (ligesom den ændrede (6) gør (4) “overflødig”).

For de særligt matematikinteresserede kommer her en grundig udledning/begrundelse for de angivne gyldighedsintervaller:

(1) Der gælder, at

$$y = \arcsin x \quad \Leftrightarrow \quad x = \sin y, \quad \begin{cases} -1 \leq x \leq 1 \\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \end{cases}$$

og at

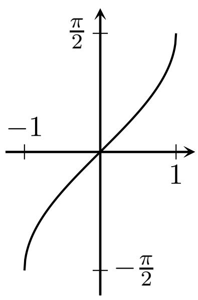
$$\tan y = \frac{\sin y}{\cos y} = \frac{\sin y}{\sqrt{1 - \sin^2 y}}, \qquad \frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\Downarrow$$

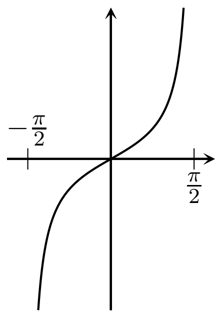
$$y = \arctan \frac{\sin y}{\sqrt{1 - \sin^2 y}}, \qquad \frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\Downarrow$$

$$\arcsin x = \arctan \frac{x}{\sqrt{1 - x^2}}, \qquad -1 < x < 1$$



Figur 1: Grafen for $\arcsin x$, $-1 \leq x \leq 1$.



Figur 2: Grafen for $\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(2) Det ses, at

$$x := \frac{x}{\sqrt{1 + x^2}} \quad \Rightarrow \quad \frac{x}{\sqrt{1 - x^2}} := \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1 - \frac{x^2}{1+x^2}}} = \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{\frac{1}{1+x^2}}} = x.$$

Indsat i (1) får vi

$$\arctan x = \arcsin \frac{x}{\sqrt{1 + x^2}}, \quad -\infty < x < \infty.$$

(3) Der gælder, at

$$y = \arccos x \quad \Leftrightarrow \quad x = \cos y, \quad \begin{cases} -1 \leq x \leq 1 \\ 0 \leq y \leq \pi \end{cases}$$

og at

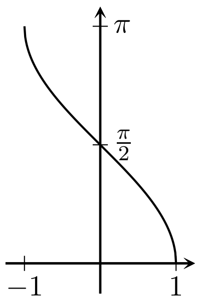
$$\tan y = \frac{\sin y}{\cos y} = \frac{\sqrt{1-\cos^2 y}}{\cos y}, \qquad \begin{cases} 0 \le y < \frac{\pi}{2} \\ \frac{\pi}{2} < y \le \pi \end{cases}$$

$$\Downarrow$$

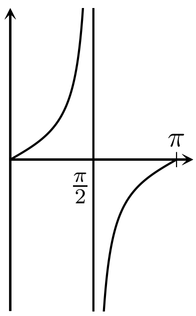
$$y = \begin{cases} \arctan \dfrac{\sqrt{1-\cos^2 y}}{\cos y}, & 0 \le y < \frac{\pi}{2} \\ \pi + \arctan \dfrac{\sqrt{1-\cos^2 y}}{\cos y}, & \frac{\pi}{2} < y \le \pi \end{cases}$$

$$\Downarrow$$

$$\arccos x = \begin{cases} \pi + \arctan \dfrac{\sqrt{1-x^2}}{x}, & -1 \le x < 0 \\ \arctan \dfrac{\sqrt{1-x^2}}{x}, & 0 < x \le 1 \end{cases}$$



Figur 3: Grafen for $\arccos x$, $-1 \le x \le 1$.



Figur 4: Grafen for $\tan x$, $0 < x < \pi$.

(4) Det ses, at

$$x := \pm\frac{1}{\sqrt{1+x^2}} \quad \Rightarrow \quad \frac{\sqrt{1-x^2}}{x} := \frac{\sqrt{1-\frac{1}{1+x^2}}}{\pm\sqrt{\frac{1}{1+x^2}}} = \frac{\sqrt{\frac{x^2}{1+x^2}}}{\pm\sqrt{\frac{1}{1+x^2}}} = \pm|x| = x.$$

Indsat i (3) får vi, at

$$\arctan x = \begin{cases} \arccos \dfrac{-1}{\sqrt{1+x^2}} - \pi, & -\infty < x \le 0 \\ \arccos \dfrac{1}{\sqrt{1+x^2}}, & 0 \le x < \infty \end{cases}$$

(5) Der gælder, at

$$y = 2\arctan x \quad \Leftrightarrow \quad x = \tan\frac{y}{2}, \qquad \begin{cases} -\infty < x < \infty \\ -\pi < y < \pi \end{cases}$$

og at

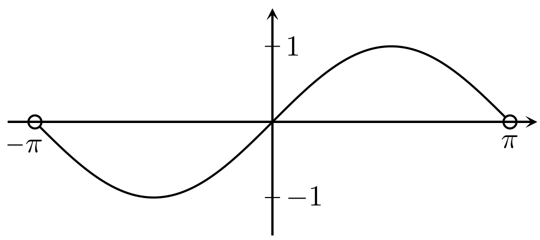
$$\sin y = \frac{2\tan\frac{y}{2}}{1+\tan^2\frac{y}{2}}, \quad -\pi < y < \pi$$

$$\Downarrow$$

$$y = \begin{cases} -\pi - \arcsin\dfrac{2\tan\frac{y}{2}}{1+\tan^2\frac{y}{2}}, & -\pi < y \le -\frac{\pi}{2} \\ \arcsin\dfrac{2\tan\frac{y}{2}}{1+\tan^2\frac{y}{2}}, & -\frac{\pi}{2} \le y \le \frac{\pi}{2} \\ \pi - \arcsin\dfrac{2\tan\frac{y}{2}}{1+\tan^2\frac{y}{2}}, & \frac{\pi}{2} \le y < \pi \end{cases}$$

$$\Downarrow$$

$$\arctan x = \begin{cases} \dfrac{1}{2}\left(-\pi - \arcsin\dfrac{2x}{1+x^2}\right), & -\infty < x \le -1 \\ \dfrac{1}{2}\arcsin\dfrac{2x}{1+x^2}, & -1 \le x \le 1 \\ \dfrac{1}{2}\left(\pi - \arcsin\dfrac{2x}{1+x^2}\right), & 1 \le x < \infty \end{cases}$$



Figur 5: Grafen for $\sin x$, $-\pi < x < \pi$.

(6) Der gælder, at

$$y = 2\arctan x \quad \Leftrightarrow \quad x = \tan\frac{y}{2}, \quad \begin{cases} -\infty < x < \infty \\ -\pi < y < \pi \end{cases}$$

og at

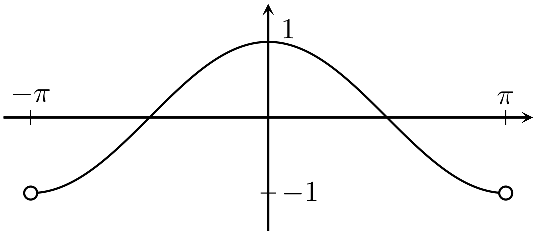
$$\cos y = \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}, \qquad -\pi < y < \pi$$

$$\Downarrow$$

$$y = \pm\arccos\frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}, \quad \begin{cases} + & \text{for } 0 \le y < \pi \\ - & \text{for } -\pi < y \le 0 \end{cases}$$

$$\Downarrow$$

$$\arctan x = \pm\frac{1}{2}\arccos\frac{1-x^2}{1+x^2}, \quad \begin{cases} + & \text{for } 0 \le x < \infty \\ - & \text{for } -\infty < y \le 0 \end{cases}$$



Figur 6: Grafen for $\cos x$, $-\pi < x < \pi$.

Formel for $\arctan \frac{1}{x}$:

$$y = \arctan \frac{1}{x}, \quad \begin{cases} -\infty < x < 0 \\ 0 < x < \infty \end{cases}$$

$$\Updownarrow$$

$$\frac{1}{x} = \tan y, \quad \begin{cases} -\frac{\pi}{2} < y < 0 \\ 0 < y < \frac{\pi}{2} \end{cases}$$

$$\Updownarrow$$

$$x = \cot y, \quad \begin{cases} -\frac{\pi}{2} < y < 0 \\ 0 < y < \frac{\pi}{2} \end{cases}$$

$$\Updownarrow$$

$$y = \begin{cases} \operatorname{arccot} x - \pi, & -\infty < x < 0 \\ \operatorname{arccot} x, & 0 < x < \infty \end{cases}$$

$$\Updownarrow$$

$$y = \begin{cases} \frac{\pi}{2} - \arctan x - \pi, & -\infty < x < 0 \\ \frac{\pi}{2} - \arctan x, & 0 < x < \infty \end{cases}$$

Altså er

$$\arctan \frac{1}{x} = \begin{cases} -\frac{\pi}{2} - \arctan x, & -\infty < x < 0 \\ \frac{\pi}{2} - \arctan x, & 0 < x < \infty \end{cases}$$

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