Exam in Discrete Mathematics

First Year at the Faculty of Engineering and Science and the Technical Faculty of IT and Design

June 4th, 2018, 9.00-13.00

This exam consists of 11 numbered pages with 14 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: ______________________

STUDENT NUMBER: SOLUTION KEY

There is only one correct answer to each question.

Problem 1 (6 %)

$P(x, y)$ means "$x \leq y$", where the domain for x and y is the set of nonnegative integers. What are the truth values of the following statements?

1. $\forall n P(0, n)$ ☒ True ☐ False
2. $\forall x \exists y P(x, y)$ ☒ True ☐ False
3. $\exists y \forall x P(x, y)$ ☐ True ☒ False

Problem 2 (8 %)

Determine whether each of the following statements is true or false.

1. If A and B are sets and $A \subseteq B$ then $|A| \leq |B|$.

   ☒ TRUE ☐ FALSE

2. If A and B are sets, A is uncountable, and $A \subseteq B$ then B is countable.

   ☐ TRUE ☒ FALSE

3. If A and B are sets with $|A| = |B|$ then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

   ☒ TRUE ☐ FALSE

4. If A is an infinite set, then it contains a countably infinite subset.

   ☒ TRUE ☐ FALSE

Problem 3 (8 %)

Determine whether each of the following theorems is true or false. Assume that $a, b, c, m \in \mathbb{Z}$ with $m > 1$

1. If $a \equiv b \pmod{m}$ and $a \equiv c \pmod{m}$ then $a \equiv b + c \pmod{m}$.

   ☐ TRUE ☒ FALSE

2. If $a \equiv b \pmod{m}$ then $2a \equiv 2b \pmod{m}$.

   ☒ TRUE ☐ FALSE

3. If $a \equiv b \pmod{m}$ then $a \equiv b \pmod{2m}$.

   ☐ TRUE ☒ FALSE

4. If $a \equiv b \pmod{m^2}$ then $a \equiv b \pmod{m}$.

   ☒ TRUE ☐ FALSE

Problem 4 (9 %)

Let $f(x) = (x! + 2^x)(x^3 + 2\log x)$, for $x > 0$. Answer the following true/false problems.

1. $f(x)$ is $O(2^x\ x^3)$ ☐ True ☒ False
2. $f(x)$ is $O(x!\ x^3)$ ☒ True ☐ False
3. $f(x)$ is $O(\log x!)$ ☐ True ☒ False
4. $f(x)$ is $\Omega(2^x\ x^3)$ ☒ True ☐ False
5. $f(x)$ is $\Omega(x!\ x^3)$ ☒ True ☐ False
6. $f(x)$ is $\Omega(\log x!)$ ☒ True ☐ False
7. $f(x)$ is $\Theta(2^x\ x^3)$ ☐ True ☒ False
8. $f(x)$ is $\Theta(x!\ x^3)$ ☒ True ☐ False
9. $f(x)$ is $\Theta(\log x!)$ ☐ True ☒ False

Problem 5 (8 %)

1. What is the value of 9^{45} **mod** 23?

   ☐ 7 ☐ 21 ☐ 2 ☒ 9 ☐ 4 ☐ 3 ☐ 6

2. What is $gcd(2^{89}, 2^{346})$?

   ☐ 2^3 ☐ 2^4 ☒ 2^{89} ☐ 3^9 ☐ 4 ☐ 2^{346} ☐ 6

Problem 6 (5 %)

Consider the following recursive algorithm:

> **procedure** *power*(n: nonnegative integer)
>   **if** $n = 0$ **then** power(n) := 3
>   **else** power(n) := power($n-1$) $\cdot$ power($n-1$)
> **return** power (n)

Which one can be calculated using this recursive algorithm?

☐ $2^n \cdot 3$ ☐ 3^{2n} ☒ 3^{2^n} ☐ $3n$ ☐ $3^n \cdot 2^{n-1}$

Problem 7 (6 %)

We consider the following moves of a particle in the xy plane

$$R : (x,y) \rightarrow (x+1,y) \text{ (one unit right)}$$
$$U : (x,y) \rightarrow (x,y+1) \text{ (one unit up) ?}$$

In how many ways can the particle move from the origin to the point (8,5)?

☐ P(8,5) ☐ C(8,5) ☒ C(13,8) ☐ P(13,5) ☐ P(13,8)

Problem 8 (6 %)

What is the binomial expansion of $(x - \frac{3}{x})^5$?

- ☐ $x^5 + 15x^3 + 90x + \frac{270}{x} + \frac{405}{x^3} + \frac{243}{x^5}$
- ☐ $x^5 + 15x^3 + 90x^2 + 270x + 405$
- ☐ $x^5 - 5x^3 + 15x - \frac{45}{x} + \frac{45}{x^3} - \frac{24}{x^5}$
- ☐ $x^5 - 15x^3 + 90x^2 - 270x + 405$
- ☒ $x^5 - 15x^3 + 90x - \frac{270}{x} + \frac{405}{x^3} - \frac{243}{x^5}$

Problem 9 (8 %)

1. What is a recurrence relation for the number of bit strings of length n that do not contain 3 consecutive 0's?
   - ☐ $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 3$
   - ☐ $a_n = 5a_{n-1} - a_{n-3}$ for $n \geq 4$
   - ☒ $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 4$
   - ☐ $a_n = 2a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 4$
   - ~~☐ $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 3$~~

2. What are the initial conditions?
   - ☒ $a_1 = 2, a_2 = 4, a_3 = 7$
   - ☐ $a_1 = 1, a_2 = 3, a_3 = 5$
   - ☐ $a_1 = 2, a_2 = 4, a_3 = 6$
   - ☐ $a_1 = 2, a_2 = 3, a_3 = 7$
   - ☐ $a_1 = 1, a_2 = 4, a_3 = 5$

Problem 10 (7 %)

Let $P(n)$ be the following statement

$$\sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}$$

We want to prove by induction that $P(n)$ is true for all $n \geq 0$.

1. What is the correct basis step of the induction proof?

   ☐ Prove that $P(1)$ is true
   ☐ Prove that $P(2)$ is true
   ☒ Prove that $P(0)$ is true
   ☐ Prove that $P(n)$ is true, for all $n \geq 1$

2. Which one of the following is a correct outline of the inductive step?

   ☐ Let $i \geq -1$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=0}^{i} 3^k = \frac{3^{i+1}-1}{2}$. Use this to prove $P(i+2)$.

   ☒ Let $i \geq 0$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=0}^{i} 3^k = \frac{3^{i+1}-1}{2}$. Use this to prove $P(i+1)$.

   ☐ Let $i = 0$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=0}^{i} 3^k = \frac{3^{i+1}-1}{2}$. Use this to prove $P(i)$.

   ☐ Let $i \geq 0$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=0}^{i} 3^k = \frac{3^{i+1}-1}{2}$. Use this to prove $P(i+2)$.

   ☐ Let $i \geq 2$ and assume that $P(i)$ is true. By the induction hypothesis $\sum_{k=0}^{i} 3^k = \frac{3^{i+1}-1}{2}$. Use this to prove $P(i+1)$.

Problem 11 (9 %)

Let $\mathcal{R}$ be the relation on the set $A = \{1, 2, 3, 4\}$ defined by

$$\mathcal{R} = \{(i, j) \mid 2 \text{ divides } (i - j)\}.$$

1. What is the matrix that represents the relation $\mathcal{R}$?

☐ $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ ☒ $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

2. Answer the following true/false problems.

$\mathcal{R}$ is reflexive	☒ True	☐ False
$\mathcal{R}$ is symmetric	☒ True	☐ False
$\mathcal{R}$ is antisymmetric	☐ True	☒ False
$\mathcal{R}$ is transitive	☒ True	☐ False
$\mathcal{R}$ is an equivalence relation	☒ True	☐ False

3. What is the matrix that represents the relation $\mathcal{R}^2$?

☒ $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

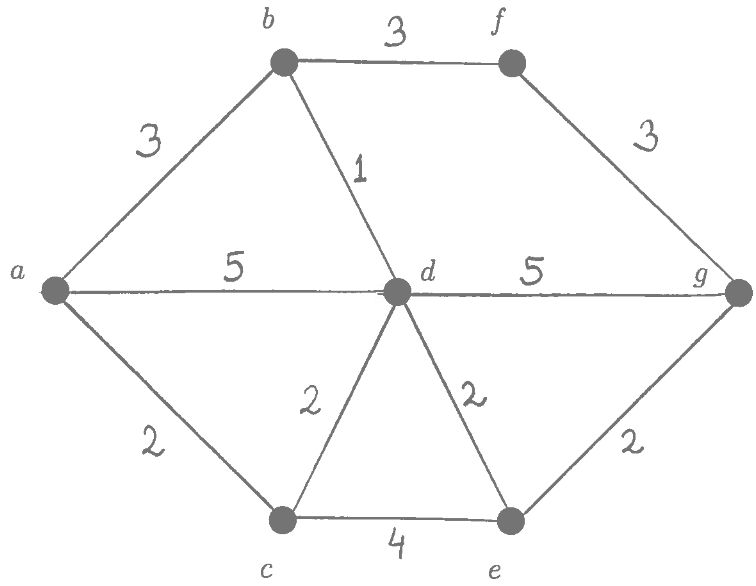

Figure 1: The graph G, considered in Problems 12, 13 and 14.

Problem 12 (8 %)

In this problem we use Dijkstra's algorithm (see Figure 2 on page 11) on the graph G in Figure 1.

1. What is the length of the shortest path from a to g (found by Dijkstra's algorithm)?

   ☐ 7 ☒ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 ☐ 13 ☐ 14

2. In what order are vertices added to the set S ?

   ☐ a, c, d, e, g
   ☐ a, b, c, f, d, e, g
   ☐ a, b, c, d, e, f, g
   ☐ a, e, b, f, g
   ☒ a, c, b, d, e, f, g
   ☐ a, e, f, b, c, g
   ☐ a, e, c, d, e, g

Problem 13 (5 %)

What is the weight of a minimum spanning tree of the graph G in Figure 1.

☐ 14 ☐ 15 ☐ 16 ☐ 17 ☒ 12 ☐ 19 ☐ 10 ☐ 13

Problem 14 (7 %)

In this problem G is the graph in Figure 1. (The edge weights of G are not considered in this problem.)

1. Answer the following true/false problems.

G has an Euler circuit	☐ True	☒ False
G has an Euler path	☐ True	☒ False
G has a Hamilton circuit	☒ True	☐ False
G has a Hamilton path	☒ True	☐ False

2. What is the length of a shortest simple circuit of G?

☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8