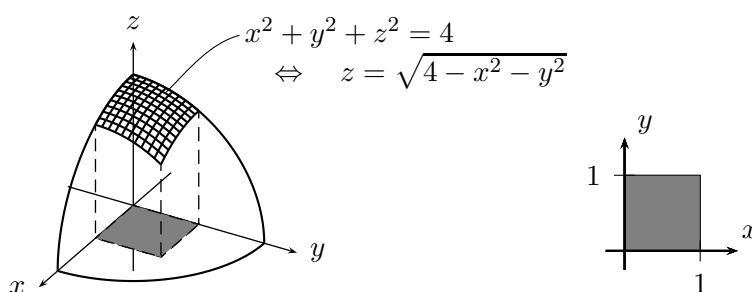


Eksempel på volumenbestemmelse

En kugle med centrum i $(0, 0, 0)$ og radius 2 gennemskæres symmetrisk af et kvadratisk prisme med sidelængde 2. Volumen af den del af kuglen, der fjernes, skal beregnes. Ved udnyttelse af symmetri om alle tre koordinatplaner kan vi nøjes med at betragte den del af voluminet, der ligger i første oktant.



$$\begin{aligned}
 V &= 8 \int_0^1 \int_0^1 \sqrt{4 - x^2 - y^2} \, dy \, dx \\
 &= 8 \int_0^1 \left[\frac{y}{2} \sqrt{4 - x^2 - y^2} + \frac{4 - x^2}{2} \arcsin \frac{y}{\sqrt{4 - x^2}} \right]_0^1 dx \\
 &= 8 \int_0^1 \left(\frac{1}{2} \sqrt{3 - x^2} + \frac{4 - x^2}{2} \arcsin \frac{1}{\sqrt{4 - x^2}} \right) dx \\
 &= 4 \int_0^1 \left(\sqrt{3 - x^2} + (4 - x^2) \arcsin \frac{1}{\sqrt{4 - x^2}} \right) dx \\
 &= 4 \left(\left[\frac{x}{2} \sqrt{3 - x^2} + \frac{3}{2} \arcsin \frac{x}{\sqrt{3}} \right]_0^1 + \int_0^1 (4 - x^2) \arctan \frac{1}{\sqrt{3 - x^2}} dx \right), \quad \text{se (1)} \\
 &= 4 \left(\left(\frac{\sqrt{2}}{2} + \frac{3}{2} \arcsin \frac{1}{\sqrt{3}} \right) + \left[\left(4x - \frac{x^3}{3} \right) \arctan \frac{1}{\sqrt{3 - x^2}} \right]_0^1 \right. \\
 &\quad \left. - \int_0^1 \left(4x - \frac{x^3}{3} \right) \frac{1}{1 + \frac{1}{3 - x^2}} \left(-\frac{1}{2} \frac{1}{(3 - x^2)\sqrt{3 - x^2}} \right) (-2x) dx \right) \\
 &= \frac{4}{3} \left(\frac{3}{2} \sqrt{2} + \frac{9}{2} \arcsin \frac{1}{\sqrt{3}} + 11 \arctan \frac{1}{\sqrt{2}} - \int_0^1 \frac{12x^2 - x^4}{(4 - x^2)\sqrt{3 - x^2}} dx \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{4}{3} \left(\frac{3}{2} \sqrt{2} + \frac{9}{2} \arctan \frac{1}{\sqrt{2}} + 11 \arctan \frac{1}{\sqrt{2}} \right. \\
 &\quad \left. - \int_0^1 \left(\frac{x^2}{\sqrt{3-x^2}} - \frac{8}{\sqrt{3-x^2}} + \frac{32}{(4-x^2)\sqrt{3-x^2}} \right) dx \right) \\
 &= \frac{4}{3} \left(\frac{3}{2} \sqrt{2} + \frac{31}{2} \arctan \frac{1}{\sqrt{2}} - \left[-\frac{x}{2} \sqrt{3-x^2} + \frac{3}{2} \arcsin \frac{x}{\sqrt{3}} \right. \right. \\
 &\quad \left. \left. - 8 \arcsin \frac{x}{\sqrt{3}} + 32 \cdot \frac{1}{2} \arctan \frac{x}{2\sqrt{3-x^2}} \right]_0^1 \right), \quad \text{se (2)} \\
 &= \frac{4}{3} \left(\frac{3}{2} \sqrt{2} + \frac{31}{2} \arctan \frac{1}{\sqrt{2}} + \frac{1}{2} \sqrt{2} + \frac{13}{2} \arcsin \frac{1}{\sqrt{3}} - 16 \arctan \frac{1}{2\sqrt{2}} \right) \\
 &= \frac{4}{3} \left(2\sqrt{2} + \frac{31}{2} \arctan \frac{1}{\sqrt{2}} + \frac{13}{2} \arctan \frac{1}{\sqrt{2}} - 16 \left(\frac{\pi}{2} - 2 \arctan \frac{1}{\sqrt{2}} \right) \right), \quad \text{se (3)} \\
 &= \frac{4}{3} \left(2\sqrt{2} - 8\pi + 54 \arctan \frac{1}{\sqrt{2}} \right) \\
 &= \frac{4}{3} \left(2\sqrt{2} - 8\pi + 54 \left(\frac{\pi}{2} - \arctan \sqrt{2} \right) \right), \quad \text{se (4)} \\
 &= \frac{4}{3} \left(2\sqrt{2} + 19\pi - 54 \arctan \sqrt{2} \right).
 \end{aligned}$$

(1): Der gælder, at

$$\arcsin \frac{1}{\sqrt{4-x^2}} = \arctan \frac{\frac{1}{\sqrt{4-x^2}}}{\sqrt{1-\frac{1}{4-x^2}}} = \arctan \frac{1}{\sqrt{4-x^2-1}} = \arctan \frac{1}{\sqrt{3-x^2}}.$$

(2): Det ses, at

$$\frac{1}{(4-x^2)\sqrt{3-x^2}} = \frac{3}{(4(3-x^2)+x^2)\sqrt{3-x^2}} = \frac{1}{2} \frac{1}{1+\left(\frac{x}{2\sqrt{3-x^2}}\right)^2} \frac{3}{2(3-x^2)\sqrt{3-x^2}}.$$

Lad $u = \frac{x}{2\sqrt{3-x^2}}$. Da er

$$\frac{du}{dx} = \frac{1}{2} \frac{\sqrt{3-x^2} \cdot 1 - x \frac{1}{2\sqrt{3-x^2}} (-2x)}{3-x^2} = \frac{3-x^2+x^2}{2(3-x^2)\sqrt{3-x^2}} = \frac{3}{2(3-x^2)\sqrt{3-x^2}}.$$

Dvs.

$$\int \frac{1}{(4-x^2)\sqrt{3-x^2}} dx = \frac{1}{2} \int \frac{1}{1+u^2} du = \frac{1}{2} \arctan u + c, \quad u = \frac{x}{2\sqrt{3-x^2}}.$$

(3): Der gælder, at

$$\begin{aligned} \arctan \frac{1}{2\sqrt{2}} &= \arcsin \frac{\frac{1}{2\sqrt{2}}}{\sqrt{1 + \left(\frac{1}{2\sqrt{2}}\right)^2}} = \arcsin \frac{1}{3} = \frac{\pi}{2} - \arccos \frac{1}{3} \\ &= \frac{\pi}{2} - \arccos \frac{1 - \left(\frac{1}{\sqrt{2}}\right)^2}{1 + \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\pi}{2} - 2 \arctan \frac{1}{\sqrt{2}}. \end{aligned}$$

(4): For $0 < x < \infty$ har vi

$$\arctan \frac{1}{x} = y \quad \Leftrightarrow \quad \frac{1}{x} = \tan y \quad \Leftrightarrow \quad x = \cot y \quad \Leftrightarrow \quad y = \operatorname{arccot} x = \frac{\pi}{2} - \arctan x.$$

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